This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

THE
CALCULUS FOR ENGINEERS
WITH APPLICATIONS TO
TECHNICAL PROBLEMS

THE CALCULUS

FOR

ENGINEERS AND PHYSICISTS.

GRIFFIN'S ENGINEERING PUBLICATIONS.

BY THE SAME AUTHOR.

MEASUREMENT CONVERSIONS (English and French). 43
Graphic Tables or Diagrams, showing at a glance the Mutual Conversion
of Measurements in Different Units of Lengths, Areas, Volumes, Weights,
Stresses, Densities, Quantities of Work, Horse Powers, Temperatures, etc.
For the use of Engineers, Surveyors, Architects, and Contractors. By Prof.
R. H. SMITH. 4to. Price 7s. 6d.

"The work is INVALUABLE "—*Colliery Guardian*.
"Ought to be in EVERY office where even occasional conversions are required.
. . . Prof. SMITH'S TABLES form very EXCELLENT CHECKS on results. . . . A VERY
USEFUL and good set of diagrams."—*Electrical Review*.

MANUALS INDISPENSABLE FOR THE ENGINEER.

By W. J. MACQUORN RANKINE, C.E., LL.D., F.R.S.,

Late Reg. Prof. of Civil Engineering in the University of Glasgow.

In Crown 8vo, Cloth.

Thoroughly Revised by W. J. MILLAR, C.E.,

Late Sec. to the Inst. of Engineers and Shipbuilders in Scotland.

RANKINE'S APPLIED MECHANICS. FOURTEENTH EDITION. 12s. 6d.
RANKINE'S CIVIL ENGINEERING. NINETEENTH EDITION. 16s.
RANKINE'S STEAM ENGINE AND OTHER PRIME MOVERS. FOURTEENTH
EDITION. 12s. 6d.
RANKINE'S MACHINERY AND MILLWORK. SEVENTH EDITION. 12s. 6d.
RANKINE'S USEFUL RULES FOR ENGINEERS. With Appendix for Electricians
by Prof. A. JAMIESON, M.I.El.E. SEVENTH EDITION. 10s. 6d.
RANKINE'S MECHANICAL TEXT-BOOK : A Simple Introduction to the Study
of Mechanics. FOURTH EDITION. 9s.
PROFESSOR RANKINE'S MISCELLANEOUS SCIENTIFIC PAPERS. With
Memoir by Prof. TAIT, M.A., and Fine Portrait on Steel. Royal 8vo,
Handsome Cloth. 31s. 6d.

APPLIED MECHANICS . . Prof. JAMIESON. 7s. 6d.
BRIDGE CONSTRUCTION . . Prof. FIDLER. 30s.
DESIGN OF STRUCTURES . . S. ANGLIN. 16s.
TRAVERSE TABLES . . . R. L. GURDEN. 21s.
MARINE ENGINEERING . . A. E. SEATON. 21s.
THE STEAM ENGINE . . . Prof. JAMIESON. 8s. 6d.
CHEMISTRY FOR ENGINEERS . BLOUNT & BLOXAM. 10s. 6d.
GAS, OIL, AND AIR-ENGINES . BRYAN DONKIN. 25s.
BOILER CONSTRUCTION . . T. W. TRAILL. 12s. 6d.
HYDRAULIC MACHINERY . . Prof. ROBINSON. 34s.
ELECTRICAL POCKET-BOOK . MUNRO & JAMIESON. 8s. 6d.
ELECTRICAL PRICE-BOOK . H. J. DOWSING. 8s. 6d.
MARINE ENGINEERS' POCKET-) SEATON AND ROUNTHWAITE.
    BOOK . . . . .)   8s. 6d.

LONDON: CHARLES GRIFFIN & Co., Ltd., Exeter St., Strand.

THE CALCULUS

FOR

ENGINEERS AND PHYSICISTS:

INTEGRATION AND DIFFERENTIATION,
WITH APPLICATIONS TO TECHNICAL PROBLEMS;

WITH

CLASSIFIED REFERENCE TABLES OF INTEGRALS
AND METHODS OF INTEGRATION.

BY

PROF. ROBERT H. SMITH,

ASSOC. M.I.C.E., M.I.M.E., M.I.EL.E., M.FED.I.MIN.E., WH. SCH., MEM. ORDER MEIJI.

WITH DIAGRAMS.

LONDON:
CHARLES GRIFFIN AND COMPANY, LTD.
EXETER STREET, STRAND.
1897.

PREFACE.

THIS work aims at the presentation of two leading features in the study and application of the higher mathematics. In the first place, the development of the *rationale* of the subject is based on essentially *concrete* conceptions, and no appeal is made to what may be termed rational imagination extending beyond the limits of man's actual physical and physiological experience. Thus no use is anywhere made of series of infinite numbers of things or of infinitely small quantities. The author believes that the logical development is both sound and complete without reference to these ideas.

In the second place, a set of Eleven Classified Tables of Integrals and Methods of Integration have been arranged in such manner as seemed best adapted to facilitate rapid reference, and thus relieve the mind engaged in practical mathematical work of the burden of memorising a great mass of formulas. This part of the work has involved very considerable labour. The germ of it is twenty-five years old in the author's manuscripts, but for the extensive and able development of it to its present form he is indebted to the co-operation of Mr R. F. Muirhead, M.A., B.Sc. (Glasgow), B.A. (Cambridge), formerly Clark Fellow of Glasgow University and Lecturer on Mathematics at Mason College.

It is hoped that these Tables may prove of great service to physicists and engineers engaged in new applications of science. In the preparation of these Tables assistance has been obtained from

many sources: among these may be mentioned, Carr's *Synopsis of the Calculus*; Legendre's Works on Elliptic Integrals and Gamma Functions; Mindung's *Tafeln der Integralen*; Sohncke's ditto; Schlömilch's Compendium; and Meyer's *Definite Integrals*.

The proofs have been read by both Mr R. F. Muirhead and the author, and it is hoped that not very many errors have escaped their repeated examinations. But the work is constructed on lines that are so novel that it cannot be hoped that it now appears without the need of some further correction. Any communication to the author giving him information of such slips will be received with sincere gratitude.

Critics who are schoolmen of the pure orthodox mathematical faith may find it hard to reconcile the ideas that have with them become innate, with some of the methods, and possibly some of the phraseology, here adopted. We only ask them to remember that there is arising a rapidly increasing army of men eagerly engaged in the development of physical research and in the industrial applications of scientific results, with the occasional help of mathematical weapons, whose mental faculties have been wholly trained by continuous contact with the hard facts of sentient experience and who find great difficulty in giving faith to any doctrine which lays its basis outside the limits of their experiential knowledge.

5th December 1896.

TABLE OF CONTENTS.

PART I.

CHAPTER I.—INTRODUCTORY.

CHAPTER II.—GENERAL IDEAS AND PRINCIPLES, ALGEBRAIC AND GRAPHIC SYMBOLISM.

CHAPTER II.—*continued.*

CHAPTER III.—EASY AND FAMILIAR EXAMPLES OF INTEGRATION AND DIFFERENTIATION.

CHAPTER IV.—IMPORTANT GENERAL LAWS.

CHAPTER V.—PARTICULAR LAWS.

CHAPTER VI.—TRANSFORMATIONS AND REDUCTIONS.

CHAPTER VII.—SUCCESSIVE DIFFERENTIATION AND MULTIPLE INTEGRATION.

CHAPTER VIII.—INDEPENDENT VARIABLES.

CHAPTER IX.—MAXIMA AND MINIMA.

CHAPTER X.—INTEGRATION OF DIFFERENTIAL EQUATIONS.

CHAPTER X.—*continued.*

PART II.

CLASSIFIED REFERENCE TABLES

OF

INTEGRALS AND METHODS OF INTEGRATION,

IN ELEVEN SECTIONS.

THE CALCULUS FOR ENGINEERS.

CHAPTER I.

INTRODUCTORY.

1. Integration more useful than Differentiation.—In phy-sical and engineering investigations the Integral Calculus lends more frequent aid than does the Differential Calculus, and the problems involving the Integral are more often of a practically important type than those requiring the Differential Calculus alone in their solution. But the ordinary student of mathematics never reaches even an elementary knowledge of Integration until he has mastered all but the most recondite portions of the science of Differentiation.

It seems *a priori* irrational, and contrary to a liberal conception of educational policy, to teach the higher mathematics in a manner so contrary to almost self-evident utility. Adherence to this the orthodox method of teaching in the Schools and Universities is, no doubt, responsible for the persistent unpopularity of this branch of knowledge and intellectual training among the classes devoted to practical work.

2. Method of the Schools.—It must be admitted that no great progress can be made in Integration without help from the results obtained by Differentiation. Therefore, so long as the two are taught as distinct subjects, by the aid of separate text-books, it is a distinct convenience to the teachers to finish off one before entering upon the other. If they be thus separated into two successive periods of study, it becomes a practical necessity to give Differ-entiation the priority in point of time.

3. Rational Method.—Still, it by no means follows that the whole of the science of Differentiation must be known before any of that of Integration can be explained, thoroughly mastered, and

A

utilised. The ordinary system of teaching the subject forces the practical student to spend on Differentiation an amount of time altogether needless for his professional objects before he enters upon Integration. Much of the former he will never use. The latter, from the very beginning, will supply him with abundant problems of immediate interest and importance in his own special work, and will, moreover, furnish him with a powerful engine that will enormously lighten the difficulties of his own professional subjects and make his progress in these tenfold more rapid.

Let it be noted, also, that very frequently the reasoning used to find an integral is essentially the same as that used to find the inverse differential. It is thus pure waste of time to go through this reasoning twice over. Once understood, it leads to the *simultaneous* recognition of the two inverse results, both of them, it may be, eminently useful. Therefore, as far as practicable, the study of Differentiation and Integration ought to be pursued *pari passu*.

4. Active Interest in the Study.—In modern education, in which such large demands are made upon the intellectual energies of the pupil, the necessity of the stimulus of a real active interest, opening out easily recognised prospects of broadening and deepening knowledge and of utilitarian advantage, ought to be conceded in the freest and most liberal fashion. Moreover, it is right to lead the pupil along the easiest road, provided it be a legitimate one. The thoroughness of the training he receives in habits of sound, trustworthy scientific thought depends more upon the length of time he is guided within the limits of correct method, and less upon whether he travels a short distance on a rugged and difficult path or a long distance upon a plainer and smoother route.

5. Object of Present Treatise.—The object of the present treatise is to introduce the student at once to the fundamental and important uses of the Integral Calculus, and incidentally to those of much of the Differential Calculus. This we desire to do in such a way as to stimulate a growing desire to progress always further in a branch of science which soon shows itself capable of supplying the key to so many practical investigations.

6. Clumsiness of Common Modes of Engineering Analysis.— At the present time our technical text-books are loaded with tedious and clumsy demonstrations of results that can be obtained "in the twinkling of an eye" by one who has grasped even only the elements of the Calculus. These demonstrations are supposed to be "elementary." They are not really so; each of them really contains, hidden with more or less skill, identically the same reasoning as that employed in establishing the Calculus formulas applicable to the case in hand. They are, in fact, simply laboured

methods of cheating the student into using the Calculus without his knowing that he is so doing. There is no good reason for this. The elements of the Calculus may be made as easy as those of Algebra or of Trigonometry. More good, useful scientific result can be obtained with less labour by the study of the Calculus than by that of any other branch of mathematics.

7. **Graphic Method.**—Much of the Calculus can be rigorously proved by the graphic method; that is, by diagram. This method is here used wherever it offers the simplest and plainest proof; but where other methods seem easier and shorter they are preferred. The present book is strictly confined to its own subject; and, wherever it is necessary, the results proved in books on Geometry, Algebra, Trigonometry, etc., are freely made use of; employing always, however, the most elementary and most generally known of these results as may be sufficient for the purpose.

8. **Illustrations.**—Everywhere the meaning and the utility of the results obtained are illustrated by applications to mechanics, thermodynamics, electrodynamics, problems in engineering design, etc., etc.

9. **Classified List of Integrals.**—The part of the book which is looked upon by its authors as the most important and the most novel is the last, namely, the Classified Reference List of Integrals. This is really a development of a Classified List of Integrals which one of the authors made twenty years ago to assist him in his theoretical investigations, and which he has found to be continuously of very great service. He has never believed in the policy of a practical man's burdening his memory with a load of theoretical formulas. Let him make sure of the correctness of these results, and of the methods by which they were reached. Let him very thoroughly understand their general meaning, and especially the limits of their range of applicability; let him recognise clearly the sort of problem towards the solution of which they are suited to help; let him practise their application to this sort of problem to an extent sufficient to make him feel sure of himself in using them in the future in the proper way. Then let him keep notes of these results in such a manner as will enable him to find them when wanted without loss of time; and let him particularly avoid wasting his brain-power by preserving them in his memory. The more brain-power is spent in memorising, the less is there left for active service in vigorous and wary application in new fields to attain new results. Formulas have a lamentable characteristic in the facility they offer for wrong application. A formula fixed perfectly in the memory, and the exact meaning and correct mode and limits of whose application are imperfectly understood, is a

pure source of misfortune to him who remembers it. It is infinitely more important to cultivate the faculty of cautious and yet ready use of formulas than to have the whole range of mathematical formulas at one's finger ends; and this is also of immensely greater importance to the practical man than to keep in mind the proofs of the formulas.

To obviate the necessity of such memorisation the "Classified Reference List of Integrals" has been constructed in the manner thought most likely to facilitate the rapid finding of whatever may be sought for. The results are not tabulated in "rational" order, that is, in the order in which one may be logically deduced from preceding ones. They are classified, firstly, according to subject, *e.g.*, Algebraical, Trigonometrical, etc., etc., and under each subject they are arranged in the order of simplicity and of most frequent utility. A somewhat detailed classification has been found desirable in order to facilitate cross-references, the free use of which greatly diminishes the bulk of the whole list. The shorter such a list is made, the easier is it to make use of.

10. Scope of Prefatory Treatise.—This treatise does not prove all the results tabulated in the "Reference List." The latter has been made as complete as was consistent with moderate bulk, and includes all that is needed for what may be described as ordinary work, that is, excluding such higher difficult work as is never attempted by engineers or by undergraduate students of physics. The treatise aims at giving a very thorough understanding of the principles and methods employed in finding the results stated concisely in the "Reference List"; proofs of all the fundamentally important results; and, above all, familiarity with the practical uses of these results, so as to give the student confidence in his own independent powers of putting them to practical use. The last chapter on the Integration of Differential Equations ought to aid greatly in pointing out the methods of dealing with various classes of problems. The ninth chapter, on Maxima and Minima, is perhaps more illustrative than any other of the great variety of very important practical problems that can be solved correctly only by the aid of the Calculus.

CHAPTER II.

GENERAL IDEAS AND PRINCIPLES—ALGEBRAIC AND GRAPHIC SYMBOLISM.

11. Meaning of a "Function."—Suppose that a section be made through a hilly bit of country for some engineering purpose, such as the making of a highway, or a railway, or a canal. The levels of the different points along the section are obtained by the use of the Engineer's Level, and the horizontal distances by one or other of the ordinary surveying methods. Let fig. 1 be the plot-
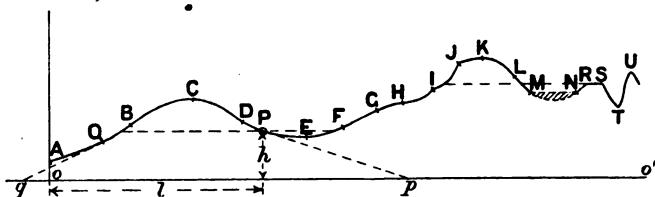
FIG. 1.

ting on paper of the section. According to ordinary practice, the heights would be plotted to a much opener scale than the horizontal distances; but in order to avoid complication in a first illustration, we will assume that in fig. 1 heights and distances are plotted to the same scale.

Each point P on this section is defined strictly by its level h and its horizontal position l. The former is measured from some conveniently chosen datum level. The latter is measured from any convenient starting-point. These two are called by mathematicians the co-ordinates of the point P on the curve ABC, etc:

For each ordinate l there is one defined value of the co-ordinate h; except throughout the stretch MN, where a break in the curve occurs. Putting aside this exception, the height h is, when this strictly definite relation exists, called in mathematical language a "function" of l; or

$$\text{Height} = \text{Function of Horizontal Distance,}$$

or, more simply written in mathematical shorthand,

$$h = \mathrm{F}(l).$$

12. Ambiguous Cases.—As seen from the dotted line drawn horizontally through P, there are three points on the section at the same level. Thus the statement that

$$\text{Distance} = \text{Function of Height}$$

or

$$l = f(h)$$

must be understood in a somewhat different sense from the first equation: namely, in the sense that, although for each height there correspond particular and exactly defined distances, still two or more such distances correspond to one and the same height, so that, if nothing but the height of a point were given, it would remain doubtful which of two or three horizontal positions it occupied. This ambiguity can only be cleared away by supplying special information concerning the point beyond that contained in the equation.

13. Inverse Functions.—The two formulas

$$h = F(l)$$

and

$$l = f(h)$$

are simply two different forms into which the relation between h and l, or the **equation to the curve**, can be thrown. The first form may be called the **solution of the equation for h**; the second the **solution of the equation for l**.

The functions $F(\ )$ and $f(\ )$ are said each to be the "inverse" of the other. An inverse function is frequently indicated by the symbol -1 put in the place of an index. For example, if s be $\sin a$, then the angle a may be written $\sin^{-1}s$. Or if l be the logarithm of a number N, or $l = \log$ N; then N $= \log^{-1}l$, which expression means that "N is the number whose log is l."

14. Indefiniteness of a Function in Special Cases.—The stretch of ground from R to S is level. Here the value of h corresponds to a continuously varying range of values of l. For this particular value of h, therefore, we have *between certain limits indefiniteness* in the solution for l.

If there were under the point J a stretch of perfectly vertical cliff, then for the one value of l to this cliff the solution for h would be similarly *indefinite* between the limits of level at the foot and at the top of the cliff.

15. Discontinuity.—From M to N there is a break in the curve. In such a case mathematicians say that h is a **discontinuous** function of l; the discontinuity ranging from M to N.

16. Maxima and Minima.—From A to C the ground rises; from C to E it falls. At C we have a summit, or a **maximum** value of h. This maximum *necessarily* comes at the end of a rising and the beginning of a falling part of the section. Evidently the converse is also true, viz., that **after** a rising and **before** the following falling part there is *necessarily* a maximum, *provided there be no discontinuity between these two parts*. There is another maximum or summit at K.

The ground falls continuously from C to E, and then rises again from E onwards. There is no discontinuity here, and E gives, therefore, a lowest or **minimum** value of h. This necessarily comes **after** a falling and **before** a rising part of the section; and between such parts there necessarily occurs a minimum, if there be no discontinuity.

We have here assumed the forward direction along the section to be from A towards the right hand. But it is indifferent whether we call this or the reverse the forward direction as regards the distinction between maximum and minimum points.

17. Gradient or Differential Coefficient.—Each small length of the section has a definite slope or **gradient**. Engineers always take as the measure of the gradient the ratio of the rise of the ground between two points near each other to the horizontal distance between, the same points. This must be carefully distinguished from the ratio of the rise to distance measured along the sloping surface. This latter is the *sine* of the angle of inclination of the surface to the horizontal; whereas the gradient is the *tangent* of the same angle of inclination. This gradient is the rate at which h increases with l. It is, in the present case, what is called a **space** rate, or length rate, or linear rate, because the increase of h is compared with the increase of a length l (not because h is a length, but because l is a length).

If at the point Q the dotted line Qq be drawn touching the section curve at Q, then the gradient at Q is the tangent of the angle QqO. The touching line at point P on the downward slope cuts OO' at p, and the tangent of PpO' is negative. It equals the tangent of PpO with sign reversed.

In the language of the Calculus this gradient is called the **Differential Coefficient of h with respect to l**. Taking the forward direction as from A towards the right hand, the gradient is **upward** or **positive** from A to the summit C; **downward** or **negative** from C to the minimum point E; positive again from E to K, and negative from K to M. From N to R it is positive, and along RS it is zero.

18. Gradients at Maxima and Minima.—At each maximum

and minimum point (C, E, K) the gradient is zero. At each maximum point (C, K) it passes through zero from positive to negative. At a minimum point (E) it passes through zero from negative to positive.

At H there is also level ground, or zero gradient. Here, however, there is neither maximum nor minimum value of *h*. This point comes between two rising parts of the section: there is a positive gradient both before and after it. Although, therefore, we find zero gradient at every maximum and at every minimum point, it is not true that we necessarily find either a maximum or a minimum wherever there is zero gradient.

19. Change of Gradient.—On the rising part of the ground it becomes gradually steeper from A to B; that is, the upward gradient increases. Otherwise expressed, there is a positive increase of gradient. From B to C, however, the steepness decreases; there is a decrease of gradient, or the variation of gradient is negative (the gradient itself being still positive).

Thus the variation of gradient being positive from A to B and negative from B to C, passes through the value zero at B, the point where the gradient itself is a maximum.

From C to D the gradient is negative, and becomes gradually steeper; that is, its negative value increases, or, otherwise expressed, its variation is negative. From D to E the gradient is negative, but its negative value is decreasing, that is, its variation is positive. Thus at D the variation, or rate of change of gradient, changes from negative to positive by passing through zero, and at this point D we have the steepest negative gradient on this whole slope CE. The steepest negative gradient, of course, means its *lowest* value. Thus at D we have a minimum value of the gradient.

20. Zero Gradients.—The distinction between the three parts C, E, and H, at all of which the gradient is zero, becomes now clear. At C the variation of the gradient is **negative**, and this gives a **maximum** height. At E this variation is **positive**, and here there is a **minimum** height. At H this rate of variation of the gradient is zero, and here, although the gradient be zero, there is neither maximum nor minimum height.

21. Discontinuity or Break of Gradient.—Wherever there is a sharp corner in the outline of the section, as at I, J, R, S, T, U, there is a **sudden** change or break of gradient. This means that at each of these points there is **discontinuity of gradient**; and the above laws will not apply to such points.

Wherever there are points of discontinuity, either in the curve itself or in its gradient, special methods must be adopted in any investigations that may be undertaken in regard to the character-

istics of the law connecting the ordinates. The methods applicable to the continuous parts of the curve may, and usually do, give erroneous results if applied to discontinuous points.

22. Infinite Gradient.—Under J the face being vertical, the gradient is commonly said to be "infinite." At each of the sharp points I, J, R, S, T, U, the variation of gradient being sudden, the rate of variation of gradient becomes "infinite." More correctly expressed, there exists no gradient at J; and at I, J, R, etc., there are no rates of variation of gradient.

23. Meaning of a "Function."—The symbolic statement

$$h = F(l)$$

is not intended to assert that the relation between l and h is expressible by any already investigated mathematical formula, whether simple or complicated. For example, in fig. 1 the said relation would be extremely difficult to express by any algebraic or trigonometric formula. Equally complicated would be the law expressing the continuous variation of, for example, the horse-power of a steam-engine on, say, a week's intermittent running; or that connecting the electric out-put of a dynamo when connected on to a circuit of variable and, perhaps, intermittent conductivity. Yet separate short ranges of these laws may in many cases be approximated to by known mathematical methods; and even when this is not possible, many very interesting, important, and practically useful special features of the general law may be investigated by mathematical means, without any exact knowledge of the full and complete law. Thus without making any reference to, or any use of, the exact form of the function $F(\ )$ in the equation applicable to fig. 1, we have already been able to point out many important features of the law it represents.

24. Horse-power as a Function of Pressure.—Again, although the actual running of, say, a steam-engine from minute to minute varies with many changes of condition, still, if we choose to investigate the separate effect of one only of these changes, for instance, change of initial pressure, it may be found fairly simple. Thus we may write

Horse-power = Function of Initial Pressure,

or

$$HP = \phi(p),$$

where p is the pressure. This means that any change of pressure changes the horse-power; and to investigate the separate effect of change of pressure on horse-power, we consider all the other con-

ditions to remain (if possible) constant, while the pressure changes. Some other conditions may themselves necessarily depend on the pressure, and these, of course, cannot be assumed to remain constant. For example, the cut-off may be supposed to remain constant. But the amount of initial steam condensation in the cylinder depends partly on the initial pressure, and it cannot, therefore, be assumed a constant in the equation $HP = \phi(p)$. Similarly, the HP may be considered as a function of the speed, it alone being varied while all other things are kept constant. Or the HP may be taken as a function of the cut-off, the initial pressure, the speed, and everything else being kept constant, while the cut-off is varied.

25. Function Symbols.—When different laws connecting certain varying quantities have to be considered at the same time, different symbols, such as

$$F(l), \ f(l), \ \phi(l), \ \psi(l),$$

are used to indicate the different functions of l referred to.

26. Choice of Letter-Symbols.—In fig. 1 we have used l to represent a distance, because it is the first letter in the word "*length,*" and similarly h to represent "*height.*" It is very desirable when letter-symbols have to be used, to use such as readily call to mind the nature of the thing symbolised. Especially in practical applications of mathematics, and more particularly when there is any degree of complication in the expressions involved, is the adherence to this rule to be strongly recommended. By keeping the mind alive to the nature of the things being dealt with, error is safeguarded against, and the true physical meaning of the mathematical operations and of their results are more easily grasped. Without a complete understanding of the physical meaning of the result, not only is the result useless to the practical man, but its correctness cannot be judged of. If, on the other hand, the physical meaning be fully grasped, any possible error that may have crept in in the mathematical process of finding the result, is likely to be detected and its source discovered without great difficulty.

27. Particular and General Symbols.—But many mathematical rules and processes have such wide application to so many entirely different physical conditions, that, in order the more clearly to demonstrate the generality of their application, mathematicians prefer to use letter-symbols chosen purposely so as to suggest only with difficulty anything endowed with special characteristics; such as x, y, z, symbols which do not suggest to the mind any idea whatever except that of absolute blankness.

It is doubtful whether this is a desirable habit in mathematical training. It seems probable that a course of reasoning might be

more firmly established in the mind of the student if he were first led through it in its concrete and particular aspect—the mind being kept riveted on one special set of concrete meanings to be attached to his symbols—and then afterwards, if need be, he may go through it again once, twice, or, if necessary, a dozen times, in order to discover (if or when this be true) that the **general form** of the result will remain the same whatever particular concrete meanings be attached to his symbols.

28. x, y, and z.—There is one feature in the use ordinarily made of x, y, z in mathematical books which the writer thinks is a real evil. In his earlier chapters the orthodox mathematician establishes a habit of using y to indicate a *function of x*: he constantly writes $y = f(x)$: that is, he takes y to represent a thing dependent on x, and which necessarily changes in quantitative value when x changes. But in his later chapters he uses y and x as two **independent variables**, that is, as two quantities having no sort of mutual dependence on each other, the variation of either one of which has no effect whatever upon the other. This is apt to, and does, produce confusion of mind ; especially as regards the true meaning of different sets of formulas very similar in appearance, one referring to y and x as mutually dependent quantities, the other referring to y and x as independent variables.

29. Functions of x.—When x is used to indicate a variable quantity, any other quantity whose value varies in a definite way with the varying values of x, may be symbolically represented in any of the following ways :—

$$F(x),\ f(x),\ \phi(x),\ \psi(x),\ \chi(x),\ \text{and X, } \mathfrak{X} \text{ or } \Xi.$$

The last forms, X, etc., are for shortness and compactness as convenient as y, and are more expressive. They will be used chiefly in connection with x in the following pages.

30. X dependent on x.—X may mean a function which is capable of being also changed by changing the values of one or more other quantities besides x ; but in so far as it is considered as a function of x, consideration of these other *possible* changes is eliminated by supposing them not to occur. This is legitimate because these other elements which go to the building up of X do not *necessarily* change with x. All elements involved in X, which necessarily change with x, are to be expressed in terms of x, and their variation is thus taken account of in calculating the variation of X.

31. Nature of Derived Functions.—In dealing with functions of this kind, mathematicians call x the "**independent variable**," a somewhat unhappy nomenclature. X and x are in physical reality mutually dependent one on the other. In the mathematical

formula, however, X being expressed in terms of x, it is considered as being derived from, or dependent upon, x; the various values of X being calculated from those of x, and the changes in X being calculated from the changes in x. Thus it should be borne in mind that the dependence of the one on the other suggested in the commonly used phrase "independent variable" is purely a matter of method of calculation, and not one of physical reality.

32. Variation of a Function.—Similarly Y may be used to indicate a derived or "**dependent**" function whose value depends only upon constants and upon the variable y.

Or L may be made to denote a derived function depending only on constants and on the variable l.

33. Scales for Graphic Symbolism.—Those readers of this treatise who are engineers must, from practice of the art of Graphic Calculation, be familiar with the device of representing quantities of all kinds by the **lengths of lines** drawn upon paper, these lengths being plotted and measured to a suitable **Scale.**

So long as the **quantity** of a function is its only characteristic with which we are concerned, each quantity can always be represented by the length of a line drawn in any position and in any direction on a sheet of paper, the scale being such that 1 inch or 1 millimetre of length represents a convenient number of units of the kind to be represented. In "Graphic Calculation" we very commonly represent on the paper also the two other characteristics of **position** and **direction** of the things dealt with; but in the Differential and Integral Calculus, so far as it is dealt with in this treatise, we are concerned *only with quantity.*

It is convenient to draw all lines representing the various values of the same kind of thing in one direction on the paper. Thus we may plot off all the x's horizontally and the corresponding X's vertically. If, when the magnitude of x is varied continuously (*i.e.*, without break or gap), the change of X be also gradual and continuous, there is obtained by this process a continuous curve which is a complete graphic representation of the law connecting X and x. The student ought at the outset to understand fully the nature of this kind of representation. It is clear that it is in its essence as wholly conventional and symbolic as is the letter-symbolism of ordinary algebra. Spoken words, written words, and written numbers are in the same way conventional; they also constitute systems of arbitrary symbolism. Graphic diagram representation is neither more nor less symbolic and arbitrary than ordinary language.

34. Ratios in Graphic Delineation.—The curve in fig. 2 is such a graphic delineation of a law of mutual dependence between X

and x. If X be of a different **kind** from x, it is impossible to form any numerical ratio between the two scales to which X and x are plotted. For instance, if the X's are ft.-lbs. and the x's feet, then the vertical scale may be, perhaps, $1'' = 10,000$ ft.-lbs., and the horizontal scale $1'' = 10$ ft.; but the number $\dfrac{10,000}{10}$, or 1000, is **not** a pure ratio between the two scales. But in physics we have relations between things of different kinds, which are called **physical ratios**. It is only by use of such physical ratios that **derived**

$$\frac{X}{T_x} = \frac{dX}{dx} = X'$$

$$\frac{X}{N_x} = \frac{dx}{dX}$$
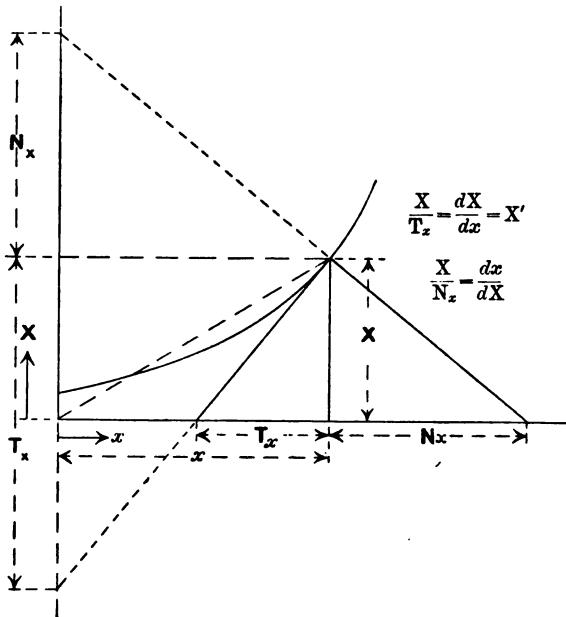
FIG. 2.

quantities are obtained. Thus the physical ratio between a number of ft.-lbs. and a number of feet, or ft.-lbs./ft., is a number of lbs. The ratio is of an altogether different **kind**, in this case lbs., from either that of the dividend or that of the divisor.

Now the ratio between a height and a horizontal distance on the paper is a gradient measured from the horizontal.

In this example, then, a gradient would mean a number of lbs.,

and each gradient would represent 6s. to a certain scale. Continuing the above example, a tangent or gradient measuring unity on the paper, *i.e.*, the tangent of 45°, would mean

$$\frac{10,000 \text{ ft.-lbs.}}{10 \text{ ft.}} = 1000 \text{ lbs.}$$

This is the scale to which gradients from axis of x are to be measured ; or

$$\text{Unit gradient} = \frac{\text{Unit height on paper}}{\text{Unit horizontal length on paper}} = 1000 \text{ lbs.}$$

Gradients measured from the axis of X have a reciprocal interpretation and are to be measured to a reciprocal scale. Thus x/X means ft./ft.-lbs., or 1/lbs., that is, the reciprocal of a number of lbs.

35. Differential Coefficient, x-Gradient or X'.—The gradient from axis of x of the curve at any point x, X, is called the "**Differential Coefficient of X with respect to x**," and is symbolised by

$$\text{either } \frac{dX}{dx} \text{ or X'.}$$

The phrase "Differential Coefficient of X with respect to x" is a cumbrous one. A shorter phrase is

the x-gradient of X ;

and as this phrase is very easily understood and definitely descriptive, it is used in this treatise.

The gradient of the curve from the axis of X is the reciprocal of the above. It is called

the Differential Coefficient of x with respect to X, or the X-gradient of x ;

and is shortly written $\frac{dx}{dX}$.

36. Scale of X'.—The scales of X and x are in general different ; and that of X' must always be different from either of these. The numerical relation between these scales and that of $\frac{dx}{dX}$ may be thus expressed :

Let

the scale of the x's be $1'' = s$ units of the x kind or quality ;

   ,,   ,,   X's ,, $1'' = S$,,   ,,   X ,,   ,,

then

the scale of the X"s is

$$\text{Unit gradient} = \tan 45° = \frac{S}{s} \text{ units of the } \frac{X}{x} \text{ kind};$$

and

the scale of the $\frac{dx}{dX}$ is

$$\text{Unit gradient} = \tan 45° = \frac{s}{S} \text{ units of the } \frac{x}{X} \text{ kind.}$$

37. Sign of X′.—The sign of X′ is + when the slope of the curve is such as to make both X and x increase positively at the
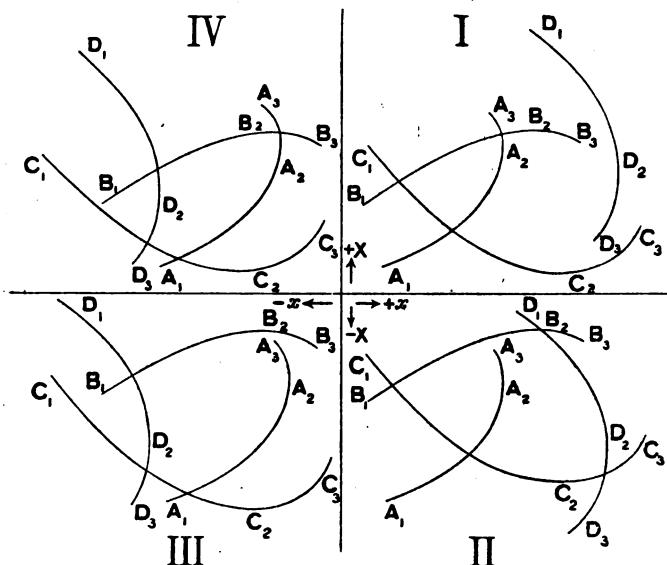
Fig. 3.

same time; it is − when it makes one increase while the other decreases. Evidently $\frac{dx}{dX}$ must always have the same sign as X′.

The possible variations of X′ and $\frac{dx}{dX}$ are very fully illustrated in fig. 3.

In fig. 3, $+x$ is measured towards the right and $+X$ upwards; negative x's are measured towards the left and negative X's downwards.

The student should follow out the variations from $+$ through 0 to $-$ of both X' and $\frac{dx}{dX}$ throughout the lengths of all the four curves A, B, C, and D in each of the four quadrants.

38. Subtangent and Subnormal.—In fig. 2 there are drawn three lines from a point xX of the curve, viz., a vertical, a tangent, and a normal. These intercept on the axis of x the lengths marked T_x and N_x on the diagram. T_x is called the subtangent and N_x the subnormal.

Since (by definition) the tangent has the same gradient as the curve at its touching point, evidently

$$X' = \frac{X}{T_x}, \text{ or } T_x = \frac{X}{X'}$$

and

$$\frac{dx}{dX} = \frac{1}{X'} = \frac{X}{N_x}, \text{ or } N_x = X'X .$$

Here T_x measured to the x-scale, and interpreted as being of the same **kind** as the x's, is a true graphic representation of X/X'.

But $\frac{dx}{dX}$ is of the same **kind** as x/X, and, therefore, would be not of the same kind as X/N_x, if N_x were measured to the x-scale, and interpreted as of the same **kind** as x. Thus in order that N_x may be used as a true graphic representation of $X'X$, which is of the same **kind** as X^2/x, care must be taken **not** to measure it to the x-scale, and **not** to interpret it as the same kind of thing as x.

If the diagram were replotted, leaving the x-scale unaltered, and making the X-scale more open, the paper-height of X would be increased, and the paper-gradient X' would be increased in the same proportion. It can easily be shown that the paper-length of T_x would remain unaltered, while that of N_x would be increased in a ratio which is the square of that in which X is increased. Similarly if, while the X-scale is maintained unchanged, the x-scale were altered so as to increase the paper-length of x, then the paper-gradient of the curve would be decreased in the same proportion as x is increased; T_x would be increased in the same proportion as x; N_x would be decreased in the same proportion.

Thus N_x in order to be a true graphic representation of $X'X$, a

quantity whose dimensions are those of X^2/x, must be measured to the scale

$$1'' = \frac{S^2}{s} \text{ units of the } \left(\frac{X^2}{x}\right) \text{ kind.}$$

In fig. 2, T_x and N_x are taken upon the x-axis, and may be termed the x-subtangent and x-subnormal. If the curve-touching line and the normal be prolonged to cut the X-axis, they and the horizontal through the touching-point will give intercepts on the X-axis, which may be termed the X-subtangent and X-subnormal, and may be written T_X and N_X. They are shown on fig. 2, and their proper scales are given below.

39. Scale of Diagram Areas.—An area enclosed by any set of lines upon such a diagram may be taken as the graphic representation of a quantity of the same kind as the product Xx, and must be measured to the scale, 1 sq. inch $= (Ss)$ units of the (Xx) kind.

40. Table of Scales.—The following is a table of interpretations of the diagram. This diagram will be constantly used hereafter for both illustrations and proofs, which latter cannot be accepted as legitimate unless the whole nature of this manner of symbolic expression be intimately understood.

TABLE OF INTERPRETATIONS AND SCALES OF DIAGRAMMATIC OR GRAPHIC REPRESENTATIONS OF DERIVATIVE AND DERIVED FUNCTIONS.

Name.	Interpretation.	Symbol.	Diagram Scale.	
Variable, ...	x	x	$1'' = s$	units of x kind.
Function of x, . .	X	X	$1'' = S$	,, X ,,
x-Gradient of X, .	Ratio of small increase of X to accompanying increase of x.	X′	$\tan 45° = \frac{S}{s}$	,, X/x ,,
X-Gradient of x, .	Ratio of small increase of x to accompanying increase of X.	$\frac{dx}{dX}$	$\tan 45° = \frac{s}{S}$	,, x/X ,,
x-Subtangent, . .	X/X′	T_x	$1'' = s$	,, x ,,
x-Subnormal, . .	XX′	N_x	$1'' = \frac{S^2}{s}$	,, X^2/x ,,
X-Subtangent, . .	$x / \frac{dx}{dX} = xX'$	T_X	$1'' = S$	,, X ,,
X-Subnormal, . .	$x / X' = x\frac{dx}{dX}$	N_X	$1'' = \frac{s^2}{S}$	,, x^2/X ,,
Area, ...	xX	A	1 sq. in. $= sS$	,, xX ,,

41. Increments.—In going forward from a point on the curve a little way, a rise occurs if the gradient be upwards. The short distance measured along the sloping curve may be resolved into two parts, one parallel to axis of x, the other parallel to axis of X. These two parts are the projections of the sloping length upon the two axes. They constitute the **differences** of the pairs of x and X co-ordinates at the beginning and the end of the short sloping length. These differences are designated by the Greek δ; thus, see fig. 4,
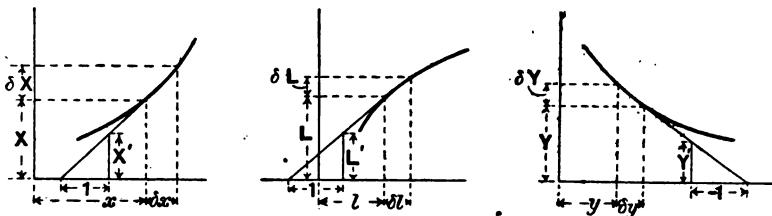
FIG. 4.

δx projection on x-axis, and
δX ,,   ,, X ,,

Since the gradient X' is the ratio of rise to horizontal distance throughout a short length, it is evident that

$$\delta X = X'\delta x.$$

If l and L be the co-ordinates, and if the gradient be called L', then this would be written

$$\delta L = L'\delta l.$$

If y and Y were the co-ordinates, the gradient being called Y', then

$$\delta Y = Y'\delta y.$$

42. Increment in Infinite Gradient.—These are the direct self-evident results of the definition of gradient, or differential co-efficient. They do not, of course, apply to points where there is no gradient, that is, to sharp corners in a diagram, where the direction of the diagram line changes **abruptly**.

If the diagram line run exactly vertical at any part, then for that part X' becomes **infinite**, and the equation appears in the form

$$\delta X = \infty \times 0$$

an indeterminate form.

This last case corresponds to the piece of vertical cliff under point J in the section fig. 1.

43. Integration.—The general case corresponds to the gradual stepping along the other parts of this section. The length of each step is projected horizontally (δl or δx) and vertically (δh or δX). The latter is the rise in level, and it equals the gradient multiplied by the horizontal projection of the length of step.

In stepping continuously from one particular point on the section to another, for instance, from A to C on fig. 1, the total horizontal distance between the two is the sum of the horizontal projections (the δl's or δx's) of all the separate steps; and the total difference of level is the sum of the vertical projections (the δh's, or δL's, or δX's) of all the separate steps. In climbing the hill, the climber rises the whole difference of level from A to C, step by step: the total ascent is the sum of all the small ascents made in all the long series of steps. If the distance be considerable, the number of steps cannot be counted, except by some counting instrument, such as a pedometer; but the total ascent remains the same, whether it be accomplished in an enormous number of extremely short steps or in an only moderately large number of long strides.

The mathematical process of calculating these sums is called **Integration.**

This mathematical process is indicated by the symbol the Greek capital Σ, when the individual steps are of definitely measurable small size. But when the method of summation employed is such that it assumes the steps to be minutely and immeasurably small, the number of them being proportionately immeasurably large, and when, therefore, of necessity the method takes no account of, and is wholly independent of, the particular minute size given to the steps, then the symbol employed is $\int$, which may be looked upon as a specialised form of the English capital S, the first letter of the word "sum." The result of the summation is called the **Integral.**

44. Increment Symbols.—The separate small portions, whose sum equals the Integral, are called the **Increments** or the **Differentials.**

When the increments are of definitely measurable small size, they are indicated by the symbols δx, δX, δh, δL, δY, etc., etc.

When they are immeasurably minute, and their number correspondingly immeasurably large, they are indicated by the symbols dx, dX, dh, dL, dY, etc., etc.

45. Integration Symbols. Limits of Integration.—The integration is carried out between particular limits, such as B and C in

fig. 1. These limits are sometimes written in connection with the symbols of integration, thus:

$$\sum_{B}^{C} \delta h \,,\; \sum_{B}^{C} \delta l \;\text{ or } \int_{B}^{C} dh \,,\; \int_{B}^{C} dl \,.$$

If $l_C h_C$ be the co-ordinates of the point C, fig. 1, and $l_B h_B$ be those of the point B, then these integrals mean the same thing as

$$(h_C - h_B) \text{ or } (l_C - l_B).$$

The limits are above indicated in the symbol by the names only of the points referred to. The points themselves are, however, frequently indicated only by the values of their co-ordinates, and then it is customary to indicate the limits of integration by writing at top and bottom of the sign of integration the limit-values of the variable whose increment appears in the intregral. Thus, since

$$\delta h = \frac{dh}{dl} \delta l$$

we have the integral of δh between B and C expressible in the two following forms :

$$h_C - h_B = \sum_{h_B}^{h_C} \delta h = \sum_{l_B}^{l_C} \frac{dh}{dl} \delta l = \int_{h_B}^{h_C} dh = \int_{l_B}^{l_C} \frac{dh}{dl} dl \,.$$

If particular points be indicated by numbers, the symbolism becomes somewhat neater. Thus the integral of δX between the points 1 and 2 of the x, X curve at which points the ordinates may be called $x_1 \, x_2$, and the co-ordinates $X_1 \, X_2$, is

$$X_2 - X_1 = \int_{X_1}^{X_2} dX = \int_{x_1}^{x_2} X' dx \,.$$

Or, again, if it were convenient to call the two limiting values of x by the letter-names a and b, then the same would appear as

$$X_b - X_a = \int_{a}^{b} X' dx \,.$$

Or, if the limiting values of x were, say, 15 and 85 feet, it would be written

$$X_{85} - X_{15} = \int_{15}^{85} X' dx \,.$$

It must be noted that the limits which are written in always refer to values of the variable whose increment or differential appears in the integration. Thus the a and b or the 85 and 15 above mean invariably values of x, not values of X nor of X'.

46. Linear Graphic Diagrams of Integration. In figs. 5 and 6 are given two methods of graphically representing this process of integration. The first corre-
sponds with the illustrations we have already employed. Here the curve xX is supposed to be built up step by step by drawing in each small stretch of horizontal length δx at a gradient equal to the known mean gradient X' for that length. The gradient X' is supposed known for each value of x, and its mean value throughout each very small length δx is therefore known.
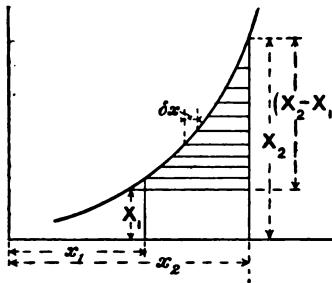
FIG. 5.

With regard to this statement it should be noted that a curve does not really possess a gradient **at a point,** but only **throughout a short length.** When we speak of the slope of a curve at a point, what we really mean is the slope of a minute portion
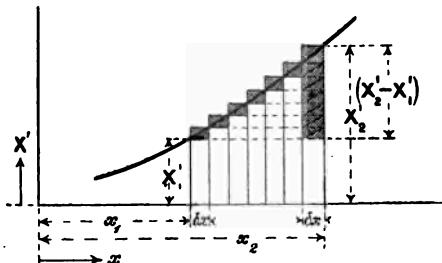
FIG. 6.

of its length lying partly in front and partly beyond the point: that is, there is actually no difference of meaning between the phrases "the slope of the curve at the point" and "the mean gradient throughout a short length at this point." Since each increment of X, or δX, equals X' times the corresponding increment of x or δx, we have in fig. 5 all these increments of X

each equal to $X'\delta x$ projected by horizontal lines on the final vertical X_2, where it is perhaps a little more plainly seen that they all sum up to $X_2 - X_1$ than without this projection, as in our previous example of climbing a hill step by step as in fig. 1.

47. **The Increment deduced from the Average Gradient.—** If this graphic method be actually employed for practical integration, then care must be taken to let the short stretch drawn at each slope X' lie partly before and partly beyond the vertical drawn at the corresponding distance x. This is necessary, because in actual graphic construction on paper we cannot make the increments δx immeasurably small. It is best to let each short curve length lie $\frac{1}{2}\delta x$ behind and $\frac{1}{2}\delta x$ beyond x.

48. **Area Graphic Diagram of Integration.—** In fig. 5 the "x-gradient," or "differential coefficient with respect to x," or X', is represented by the tangent of an angle on the paper.

In fig. 6 the x's are again represented by horizontal lines, and the differential coefficients are also represented by vertical lines, while the integral is represented by an area. The known values of X' for successive values of x are plotted upwards co-ordinate with the x's, and a curve is thus obtained. The area of the vertical strip, bounded at the top by the curve, at its sides by two vertical straight lines δx apart, and at its foot by the axis of x, equals the product $X'\delta x$, where X' is the mean value throughout the length δx. The successive strips thus formed are contiguous to each other, so that the sum of them, that is, the integral $\int X'dx$, is the total area underneath the curve and above the axis of x included between the two verticals at x_1 and x_2, the limiting values of x.

Here, again, an error would be made if, instead of taking the mean value of X' for each length δx, there were taken either its value at the beginning of δx or that at the end of δx. If for each δx the value of X' at its beginning were used, the total area, or integral, obtained would be too small by the sum of the small triangular areas rafined as in the lower parts of the rectangles in fig. 6. If, on the other hand, the value at the end of each δx were used, the calculated integral would be greater than the true integral by the sum of the small triangular areas rafined as in the upper parts of the rectangles in fig. 6. This is clearly true whether the δx's be uniform in size or of differing sizes.

49. **Diminishing Error.—** Not only does the size of each such error for each individual δx diminish indefinitely as δx is made smaller, but their sum also diminishes indefinitely in spite of their number being indefinitely increased. Although this is not really a

proposition of any special theoretical importance, since it is always the average, and not either the initial or final values of X′ that are to be used, still it is interesting and customary to prove it to be true. The neatest proof is that given by Newton, and is explained by fig. 6. For the purposes of the proof all the δx's are assumed to be of one uniform size, and, therefore, each of them equal to the last before x_2 is reached. This being so, all the above-mentioned triangular area-errors are transferred horizontally so as to lie between the vertical x_2 and $(x_2 - \delta x)$. The sum of each set of errors is now easily seen to be equal to about $\frac{1}{2}(X′_2 - X′_1)\delta x$, the sum of both sets taken together being exactly equal to this last strip. Now the height $(X′_2 - X′_1)$ of this strip has a definite finite known value ; and, therefore, the area $(X′_2 - X′_1)\delta x$ diminishes always in size as δx is made smaller.

It must be clearly understood that no such error, large or small, occurs if the X″s employed are the true average heights of the narrow strips. The average height of the strip is found at the middle of its breadth when the upper boundary is straight, and deviates from the middle by only a minute distance, if the upper boundary be very nearly straight, as it does become when its length is made very short.

50. **Integration through Infinite Gradient.**—A simple consideration of the vertical cliff below J, in fig. 1, where the gradient is **infinite**, will show that, while the process of integration written

$\int \dfrac{dh}{dl} \cdot dl$ is possible up to the base-point of the cliff, and also possible

from the top edge of the same onwards, it is impossible throughout the height of the cliff, simply because it has no meaning. Here neither the gradient nor the increment δl (or δx) has any existence. Mathematicians so constantly speak of the infinitely great and the infinitely small, or **infinity** and **zero**, that probably many novices in mathematical reasoning fall into the habit of imagining that there are really such things. But neither has any sort of existence except as symbols, and in this case the symbols correspond to no physical fact, and to no mental conception. Throughout the vertical cliff there is no horizontal distance, i.e., there is no δl ; and there is no gradient, because a gradient is the ratio between the vertical and horizontal projections of a sloping length, and since there is here no horizontal projection, there can be no ratio between the two.

The phrases "infinity" and "zero" are merely word-symbols to indicate the directions in which very large things and very small things vanish beyond the range of our perceptions, as they grow larger and smaller.

There is, however, nothing meaningless or impossible in $\int d\mathrm{X}$ at the same place. In fig. 1 up the vertical face under J, the $\delta\mathrm{X}$'s, or δh's, have the same concrete, finite meaning as they have elsewhere. Thus it is clearly improper to write $\int d\mathrm{X} = \int \mathrm{X}' dx$ for this part of the integration; the formula, which is *true in general*, fails under these special conditions.

51. Change of Form of Integral.—If L be a function of the variable l, and if its l-gradient be called L', then $\delta \mathrm{L} = \mathrm{L}' \delta l$; and if λ be any other function of l, then $\int \lambda dl = \int \dfrac{\lambda}{\mathrm{L}'} d\mathrm{L}$. When λ and L' are both capable of simple expression in terms of L, the latter form of the integral may be more easily dealt with than the former. Such a transformation of an integral is called a **change of the independent variable** or "**substitution.**" *

52. Definite and Indefinite Integrals.—Sometimes the limits of the integration are not expressed in the written symbol, which then stands simply $\int \mathrm{X}' dx$. When thus written, it is understood that in the integration the variable x increases continuously up to an **undefined** limiting value, which is to be written x in the expanded form of the integral. In fact by $\int \mathrm{X}' dx$ is meant $\int^x \mathrm{X}' dx$, the upper limit being **any** final value of the gradually increasing x.

The lower limit may be written without defining the upper limit. Thus $\int_a \mathrm{X}' dx$ means $(\mathrm{X} - \mathrm{X}_a)$. If various upper limiting values of x be successively taken, the part of the integral function involving a remains unchanged.

Such an integral may be written

$$\int \mathrm{X}' dx = \mathrm{X} + \mathrm{C},$$

that is, as the sum of two terms, one of which, C, remains unchanged when the upper limit is varied, while the other, X, remains the same although the lower limit be changed. This is called the **indefinite integral**, and C is called the **constant of integration.**

When both upper and lower limits are particularised, as in $\int_a^b \mathrm{X}' dx$, the quantity is called a **definite integral.**

53. Integration Constant.—To show the exact meaning of the

* See Classified List, II. G.

integration constant C, compare the above two forms of writing the indefinite integral. The values of X being the same in both cases, it is clear that C equals $(-X_a)$. The integration constant, therefore, depends on the implied lower limit of x $(=a)$. If C be given, the implied lower limit a is thereby fixed; and conversely, if a be given, its value determines that also of C.

The indefiniteness of the indefinite integral may, therefore, be considered as due to free choice being left as to either or both limits. The part X depends on the choice of the upper limit, and remains indefinite so long as that is not fixed. The part C depends on the lower limit, and is indefinite until this limit is fixed.

54. Meaning of Integration Constant.—Figs. 7 and 8 may help to elucidate further this question of limits and of integration constant. In fig. 7 the same curve is drawn thrice in different positions in the diagram. P'Q'R' is PQR simply raised at every point
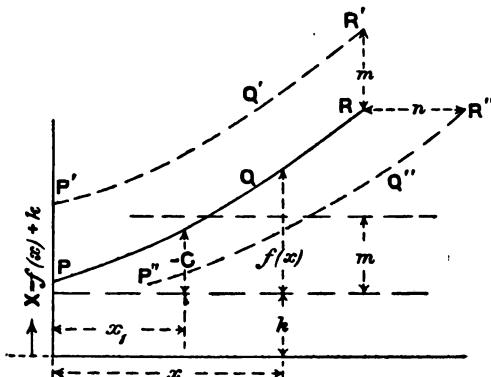
FIG. 7.

through the height m. P"Q"R" is the same as PQR shifted horizontally the distance n. Since X' is the gradient of the curve, the same values of the integral $\int X' dx$ will be obtained from all three curves if it be taken between limits on each which give the same series of values of X'. Thus the integrals will be the same when obtained from PQR and from P'Q'R' if the same limits of x be used in each case. They will be the same from PQR and from P"Q"R" if the same limits of X be used, which will mean limiting values of x in P"Q"R" greater by n than those in PQR.

These upward and right-hand horizontal shiftings of the curve

are. equivalent to equal downward and left-hand horizontal shift-
ings of the axes from which the co-ordinates X and x are measured.
Thus the two shiftings are combined in fig. 8.   Here, in order to
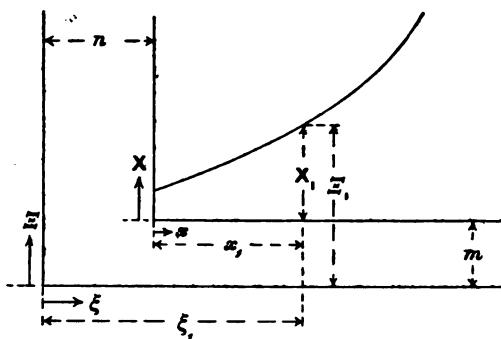
FIG. 8.

obtain the same series of values of X′, and thus the same value of

the integral $\int X'dx$ when the co-ordinates are measured from one set

of axes as when they are measured from the other, both upper and
lower limiting values of x must be greater by n when measured
from one set of axes than when measured from the other set; and
this will mean also that the two limiting values of X are both
greater by m for the one set of axes than for the other.
    It is thus clear that the limits of a definite integral refer to
defined points on the curve at which occur particular stated
gradients (X′), that is, values of the function to be integrated;
and *not* to points with particular stated values of either the variable
(x) or of the integral (X).   Since the function to be integrated (X′)
is stated in terms of the variable (x), therefore the desired limits of
X′ are conveniently named by reference to the corresponding
values of x; but these latter really depend upon the arbitrarily
chosen axes of reference.
    55. Meaning of Integration Constant. — In the indefinite

integral $\int X'dx = X + C$, if the part X is understood to mean the

whole integral function at the undefined upper limit and measured
from an arbitrarily chosen datum level, then X will include some
constant such as m in figs. 7 and 8.   But if X be understood to
mean the sum of those terms of the integral function which involve
x, excluding all terms in which x does not appear; then X will

have a value independent of the arbitrary datum level from which X is measured (so that m will not appear): but, on the other hand, terms involving n will appear, *i.e.*, the value of X will depend on the position of the axis from which the x's are measured.

56. Extension of Meaning of Integration.—In the expression $X + C$, taking X to represent the whole integral quantity, it is often easy to select the axes of reference so as to give a specially convenient value to C.

For example, if the integral X be the deflection of a symmetrically designed girder, symmetrically loaded and supported at its two ends, the symmetry of the problem makes it *a priori* evident that equal deflections occur at equal distances on either side of the centre of the girder length; and, if this centre be chosen as the origin from which to measure the x's, equal deflections will be found for equal $+$ and $-$ values of x, and this condition, if applied, will be sufficient to determine one integration constant.

57. Integration the Inverse of Differentiation.—It is important to recognise that the operation of integrating a known function X' can lead to only one definite result, or rather to a series of results which differ from each other only in the value of the constant C.

If through any point of $P'Q'R'$ in fig. 7 there were drawn any curve deviating at any part from $P'Q'R'$, it is evident that, at some parts at least of this other curve, it would have a gradient different from that of the part of PQR lying immediately below or above it. Thus all curves having the same X' for each x are related to each other as are PQR and $P'Q'R'$. Now these two curves taken between the same limits give the same difference of height; and, if the integral be taken in the "indefinite" form, as meaning the total height at any x from the horizontal axis, the heights of the two curves differ everywhere by the same amount m, that is, the two indefinite integrals differ only in the value of the constant C.

58. Usual Method of finding New Integrals.—Since, then, a definite function of x when integrated with respect to x gives one and one only function of x (definite in all but the particular value of the additive constant), therefore, if we happen to know of a curve which gives at each point an x-gradient equal to the function of x which we wish to integrate, we conclude with certainty that the height of that curve is the integral sought for. The level of the axis from which the height is to be measured is fixed by the special, or "limiting," conditions of the problem.

This is the usual method of finding integrals: namely, we make use of our previous knowledge of differentials. The finding of differential coefficients, or gradients, is an easier process in general than the reverse or inverse process of finding integrals. This differ-

ence is analogous to that we observe in ordinary geometry, in which it is simpler to find the tangent of a known curve than it is to find the curve by building it up from a knowledge of the direction of its tangent at each point.

In some simple cases, examples of which will be given in the next chapter, it is as easy to find the integral directly as it is to find the differential. Very often the study of the character of a curve or law leads to the simultaneous recognition of the differential and the inverse integral.

CHAPTER III.

EASY AND FAMILIAR EXAMPLES OF INTEGRATION AND DIFFERENTIATION.

59. Circular Sector.—An easy example of integration is the finding the area of a circle, or of any sectorial part of a circle.

In fig. 9 there is drawn an isosceles triangle of very small vertical angle placed at O, the centre of a circle, and whose correspondingly small base is a very short tangent to the arc of the circle.
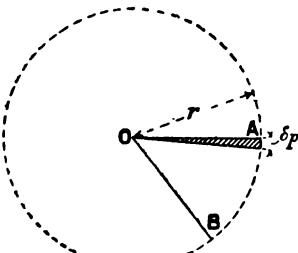
FIG. 9.

The height of this triangle is r, the radius of the circle. Its base is, in a minute degree, longer than the arc, but equals the arc in length with an approximation which is closer as the vertical angle at O becomes smaller. Taking δp, the peripheral arc length, as an approximation to the base, the area is $\frac{1}{2}r.\delta p$. The whole angle AOP may be split up into a very large number of minute angles, in each of which may be formed a small triangle similar to the above. The sum of the areas of these triangles is greater than that of one of them in the same ratio that the sum of their bases is greater than the base of one of them, because the factor $\frac{1}{2}r$ has the same value in all. The series of bases form a connected chain of very short tangents lying outside the arc AB. As the individual links of this chain become shorter, and the total number of them correspondingly greater, the sum of their lengths becomes equal to the arc length AB with closer and closer approximation, and, at the same time, the sum of the triangular areas

equals that of the circular sector ABO with closer and closer approximation. Thus, taking the arcs minutely short, and calling the arc length AB by the letter p, we find

$$\text{Circular sectorial area} = \tfrac{1}{2}r.p \, .$$

For the complete circular area the length of arc p becomes the complete circumference of the circle or $2\pi r$. Thus the complete circular area is πr^2.

Here the important point to note is, that the approximation of the sum of the triangular areas to equality with the circular area proceeds *pari passu* with that of the sum of the tangent lengths to the circular arc length.

60. Constant Gradient.—In this integration the factor $\tfrac{1}{2}r$ is a constant. The variable is p, the arc length measured from some given point on the circular circumference. The increment of p is δp: that is, δp may be looked upon as the increase of the arc length p, as the sectorial area is swept out by a radius revolving round the centre O. The integral is the gradually increasing area so swept out. The increment, or differential of this area, is $\tfrac{1}{2}r.\delta p$. Thus $\tfrac{1}{2}r$ is the differential coefficient of the area with respect to p; or $\tfrac{1}{2}r$ is the p-gradient of the area. Calling the constant $\tfrac{1}{2}r$ by the letter k, we may write this

$$\frac{d}{dp}(kp) = k \, ; \text{ or otherwise } \int k\,dp = kp \, .^*$$

61. Area of Expanding Circle.—A second simple example of integration is that of the area swept through by the circumference of a gradually expanding circle. Let the radius from which the expansion begins be called r_1: the area inside this initial circumference is πr_1^2. At any stage of the expansion when the radius has become any size r, the area swept through has been $(\pi r^2 - \pi r_1^2)$. This is the indefinite integral; the constant of integration being here $-\pi r_1^2$.

As r increases by δr from $(r - \tfrac{1}{2}\delta r)$ to $(r + \tfrac{1}{2}\delta r)$, the area swept out by the circumference is a narrow annular strip whose mean peripheral length is $2\pi r$, and whose radial width is δr, as shown on fig. 10. The area of this annular strip, therefore, equals $2\pi r.\delta r$, and this is the increment, or differential, of the integral area swept out. The "differential coefficient with respect to r," or the "r-gradient," of the area is, therefore, $2\pi r$. That is, the r-gradient of $(\pi r^2 + C)$, where C is any constant, is $2\pi r$. Otherwise written,

$$\frac{d}{dr}(\pi r^2 + C) = 2\pi r \, .$$

* See Classified List, I. 4, and III. A. 1.

Here π is a constant multiplier, and, if any other constant multiplier k were used, the result would be

$$\frac{d}{dr}(kr^2 + C) = 2kr.$$

Written in the inverse manner, the same result appears as

$$\int 2krdr = kr^2 + C.*$$

62. Rectangular Area.—A third simple example is shown in fig. 11. Here a vertical line of length k is moved horizontally to the right from an initial position x_1. Its lower end moves along the axis of x. Its upper end moves along a horizontal straight

FIG. 11.

line at the height k above this axis. The vertical sweeps out a rectangular area equal to k multiplied by the length of horizontal movement. At any stage x of the movement the area swept out

* See Classified List, III. A. 2.

is $k(x - x_1)$. Here the variable is x. Call its increment δx. While the vertical moves horizontally δx, the area swept out is a narrow rectangular strip equal to $k\delta x$. This is the increment of the integral area. Thus the x-gradient of the area is k. Calling the constant $-kx_1$ by the letter C, this result may be written in the two forms

$$\frac{d}{dx}(kx + C) = k$$

and

$$\int k\, dx = kx + C \,.^{*}$$

This result is identical in form with that of § 60, fig. 9, except that in the latter the integration constant is zero and does not appear, and the variable is named p, while x is the name adopted in fig. 11. The choosing of one or other name to indicate the variable is, of course, of no consequence; but taken as two illustrations of purely geometric integration, the two results are of *different kinds*, although taken as formulas simply they are identical.

63. **Triangular Area.**—In fig. 12 we have an area swept out by a vertical line as it moves from left to right, and increases in height at a uniform rate during the motion. Its lower end lies always in the axis of x. Its upper end lies in a straight line whose inclination to the x-axis is called m.

FIG. 12.

During the movement δx, from $(x - \frac{1}{2}\delta x)$ to $(x + \frac{1}{2}\delta x)$, the mean height of the small strip of area swept out is $(k + mx)$, and its area, therefore, is

$$(k + mx)\delta x \,.$$

The whole area swept out during the motion from the lower limit x_1 to any upper limit x may be divided into two parts; one rectangular with the base $(x - x_1)$ and the height $(k + mx_1)$; the other triangular with the same base and the height $m(x - x_1)$.

* See Classified List, III. A. 1.

This whole area is, therefore,

$$(x - x_1)\{(k + mx_1) + \tfrac{1}{2}m(x - x_1)\} = k(x - x_1) + \tfrac{1}{2}m(x^2 - x_1^2).$$

This may be written in the form

$$\left[kx + \tfrac{1}{2}mx^2 \right]_{x_1}^{x}$$

which form means that the function of x standing inside the square bracket is to be calculated for the two values of x, x and x_1 (the former "indefinite" or not particularised, the latter special), and the latter value of the function subtracted from the former.*

Again, it may be written in the other form

$$kx + \tfrac{1}{2}mx^2 + C$$

where C is a "constant of integration."

This result may be expressed by either of the two following equations :—

$$\frac{d}{dx}\left\{ kx + \tfrac{1}{2}mx^2 + C \right\} = k + mx$$

and

$$\int (k + mx)dx = kx + \tfrac{1}{2}mx^2 + C.$$

64. First and Second Powers of Variable.—The last integral may be split into two parts. The first is

$$\int k\,dx = kx + C_1$$

which is identical with what is obtained in fig. 11. The second is

$$\int mx\,dx = \tfrac{1}{2}mx^2 + C_2$$

which is the sweeping out of the triangular area.

65. Integral Momentum.—The following are other easy examples of the first of these two formulas.

The extra momentum acquired by a mass m in the interval between time t_1 and time t_2, during which its velocity is accelerated at the constant rate g, if its velocity be v_1 at time t_1, is

$$m(v_2 - v_1) = \int_{t_1}^{t_2} mg\,dt = mg(t_2 - t_1).$$

* See Classified List, " Notation."

Here mg is the acceleration of momentum, or the time-gradient of the momentum.

66. Integral Kinetic Energy.—The simultaneous increase of Kinetic Energy is

$$\frac{m}{2}(v_2{}^2 - v_1{}^2) = \frac{m}{2}\{g^2(t_2 - t_1)^2 + 2g(t_2 - t_1)v_1\}$$

$$= mg(t_2 - t_1)\{v_1 + \tfrac{1}{2}g(t_2 - t_1)\}$$

$$= \text{Extra Acquired momentum} \times \text{Average Velocity during interval.}$$

67. Motion integrated for Velocity and Time.—Again the distance travelled by a train between the times t_1 and t_2, when running at a constant velocity v, is

$$\int_{t_1}^{t_2} v dt = v(t_2 - t_1).$$

Here the velocity v is the time-gradient of the distance travelled.

68. Motion from Acceleration and Time.—Easy examples of the second formula are the following :—

If the velocity of a mass be accelerated at the uniform rate g; then, since the velocity at any time t is $\{g(t - t_1) + v_1\}$, and since in a small interval of time δt, the distance travelled is $v.\delta t$, where v is the average velocity during δt, we find the distance travelled in interval $(t_2 - t_1)$ to be

$$\int_{t_1}^{t_2} \{gt - gt_1 + v_1\}dt = \frac{g}{2}(t_2{}^2 - t_1{}^2) - gt_1(t_2 - t_1) + v_1(t_2 - t_1)$$

$$= (t_2 - t_1)\{v_1 + \tfrac{1}{2}g(t_2 - t_1)\}.$$

If this be multiplied by mg, we get again the increase of kinetic energy as shown above in § 66; so that the increase of kinetic energy equals the uniform acceleration of momentum (mg) multiplied by the distance travelled.

69. Bending Moments.—As another example, take a horizontal beam loaded uniformly with a load w per foot length. If we name by the letter l lengths along the beam from any section where we wish to find the bending moment due to this load; then on any short length δl there is a load $w.\delta l$, and the moment of this load upon the given section is $wl.\delta l$, where l means the length to the

C

middle of δl. The integral, or total, moment exerted upon this section by the part of the load lying between l_1 and l_2 is

$$\int_{l_1}^{l_2} wl\, dl = \frac{w}{2}(l_2^2 - l_1^2)$$

$$= w(l_2 - l_1) \times \tfrac{1}{2}(l_2 + l_1)$$

= whole load on $(l_2 - l_1)$ multiplied by the distance of the
middle of the same length from the given section.

It must be noted that this is the moment exerted by the load alone independently of that exerted by the forces supporting the beam.

70. Volume of Sphere.—Passing now to volumetric integrals, we may consider a very small sectorial part of the volume of a sphere as an equal-sided cone of very small vertical angle placed at the centre of the sphere, and with a very small spherical base nearly coinciding with the flat surface of small area touching the sphere. The volume of the small cone with the flat base is known to be $\frac{1}{3}$ the product of its base area by its height. The height here is r, the radius of the sphere. This is true whatever be the shape of the cross section of the cone. Now the whole volume of the sphere is made up of a very large number of such small-angled cones **with spherical bases**, these cones fitting close together so as to fill up the whole space. They would not fit close together if their cross sections were, say, circular; but the argument does not depend on the shape of the cross section, and this is to be taken such as will make the cones fit close together. In all these small conic volumes, the common factor $\frac{1}{3}r$ appears as a constant: each is $\frac{1}{3}r.\delta A$, if δA represent the area of the small base. Thus the sum of the volumes is greater than any one of them in the same ratio as the sum of the areas of the bases is greater than the base-area of that one. Thus if A be the sum of the bases, or $\int dA = A$, we have the sum of the volumes equal to $\frac{1}{3}rA$. For any sectorial portion of the volume of the sphere, the sum of the areas of the flat tangent bases approximates to the area of the corresponding portion of the spherical surface *pari passu* with the approximation of the sum of the flat-based conical volumes to the sum of the round-based conical volumes, which latter is the true spherical volume. Thus, if A be the area of the spherical surface, the volume subtended by it at the centre is $\frac{1}{3}rA$. If A be taken as the complete spherical surface, then $\frac{1}{3}rA$ is the total spherical volume. This integration is in form identical with that of fig. 9. It differs from that in **kind**, inasmuch as the differential δA is an

area, while in fig. 9 the differential δp is a line. The mathematical process is the same in both cases; but the legitimacy of the application of this process depends in the one case upon the physical relations between certain curved and straight lines, while in the other case it depends on the physical relations between certain curved and flat surfaces.

When it is known that the ratio of the surface-area of a sphere to the square of its radius is 4π, the above integration proves the complete spherical volume to be $\frac{4}{3}\pi r^3$ (see § 76 below).

71. **Volume of Expanding Sphere.**—Consider now the spherical volume as swept through by the surface of a gradually expanding sphere. If the radius be r_1 at one stage of the expansion, and r at another, the volume swept through between these two stages is $\frac{4}{3}\pi(r^3 - r_1^3)$. During any small increase of size δr from the radius $(r - \frac{1}{2}\delta r)$ to $(r + \frac{1}{2}\delta r)$, the volume swept out is the normal distance δr between the smaller and larger spherical surfaces multiplied by the mean area of the spherical surface during the motion, viz., $4\pi r^2$. That is, the increment of volume is

$$4\pi r^2 \delta r.$$

The definite integral of this is, as above stated,

$$\left[\tfrac{4}{3}\pi r^3 \right]^r_{r_1}$$

and the **indefinite** integral for an indefinite size r is

$$\tfrac{4}{3}\pi r^3 + C.$$

Thus $4\pi r^2$ is the r-gradient of $(\frac{4}{3}\pi r^3 + C)$.

If x were used to represent the radius, and X the volume, and X' the x-gradient of X and the constant factor 4π be written k: we would here have

$$X' = 4\pi x^2 = kx^2$$

$$X = \int kx^2 dx = \frac{k}{3}x^3 + C.^*$$

Expressed in words, the radius-gradient of the spherical volume is the spherical surface.

72. **Volume of Expanding Pyramid.**—Consider a rectangular-based pyramid of height x, and the two sides of whose base are mx and nx. The area of the base is mnx^2, and, therefore, the pyramidal volume is $\frac{1}{3}mnx^3$. Now, suppose the size of this pyramid to be gradually increased, keeping its shape unaltered, by extending

---

* See Classified List, III. A. 2.

its sides in the same planes, and moving the base away from the vertex while keeping the base always parallel to its original position. As the height x increases, the sides of the rectangular base both increase in the same ratio so as to remain always mx and nx; and, therefore, the increasing volume is always equal to $\frac{1}{3}mnx^3$. As the base moves a distance δx away from the vertex from the height $(x - \frac{1}{2}\delta x)$ to $(x + \frac{1}{2}\delta x)$, the increase of volume thus added to the pyramid is the mean area of the base during this motion, viz., mnx^2, multiplied by the normal distance δx between the old and the new bases. The increment of volume is thus $mnx^2.\delta x$. The definite integral volume taken between the limit x_1 and x_2 is

$$\left[\tfrac{1}{3}mnx^3\right]_{x_1}^{x_2}.$$

If the constant factor mn be written k, this result would be thus expressed, taking the indefinite form of the integral :—

$$\int kx^2dx = \tfrac{1}{3}kx^3 + \mathrm{C},$$

which is formally or symbolically identical with the last result obtained. The difference between the two in **kind** is perhaps best recognised by comparing the word-expression of the last result with the following similar statement of our present one :—

> The height-gradient of the volume of a pyramid of given shape is the area of the base of the pyramid.

In this last statement of the result no reference is made to the special shape of the cross section of the pyramid, and it is readily perceived that the reasoning employed above did not depend in any degree upon the rectangularity of the base.

73. **Stress Bending Moment on Beam.**—Take as another example of this formula leading from the second power in the gradient to the third power in the integral, the calculation of the stress-bending moment of a rectangular beam section exposed to pure bending of such degree as produces only stresses within the elastic limit. Under this condition the normal stress on the section increases uniformly with the distance from the neutral axis, which in this case is at the middle of the depth. Thus, if the whole depth of the section be called H, and the intensity of stress at the top edge (at distance $\dfrac{\mathrm{H}}{2}$ from neutral axis) be called k; then the intensity of stress at any distance h from the axis is $k\,\dfrac{h}{\frac{1}{2}\mathrm{H}} = \dfrac{2k}{\mathrm{H}}\,h$.

If the width of the section be B, the area of a small cross strip of it, of depth δh, is $B\delta h$. If h mean the height to the middle of δh, then the whole normal stress on this strip is $\frac{2k\text{B}}{\text{H}} \cdot h\delta h$, and the moment of this round the neutral axis is $\frac{2k\text{B}}{\text{H}} h^2 . \delta h$, because h is the leverage. The sum of these moments over the half of the section lying above the axis is the integral of this between the limits $h = 0$ and $h = \frac{1}{2}\text{H}$, or

$$\int_0^{\frac{1}{2}\text{H}} \frac{2k\text{B}}{\text{H}} h^2 dh = \left[\frac{2k\text{B}}{3\text{H}} h^3\right]_0^{\frac{\text{H}}{2}}$$

$$= \frac{2k\text{B}}{3\text{H}} \cdot \frac{\text{H}^3}{8}$$

$$= \frac{k}{12} \text{BH}^2 .$$

An equal sum of moments of like sign is exerted by the stresses on the lower half of the section, and thus the

<div align="center">Total Stress Bending Moment $= \frac{1}{6}k\text{BH}^2$.</div>

74. Angle Gradients of Sine and Cosine and Integration of Sine and Cosine.—In fig. 13 the angle a is supposed measured in radians, that is, in circular measure, the unit of which is the angle whose arc equals the radius. Radians, sines, cosines, tangents, etc., are pure numbers, or ratios between certain lengths and the radius of a circle; but if the radius be taken as unity, as in fig. 13, then these ratios are properly represented by lengths of lines, this graphic representation being to an artificial scale just as to other artificial scales, velocities, moments, weights, etc., can be graphically represented by line-lengths. In fig. 13 the angle a is measured to such a scale by the length of the arc Na, while to the same scale sin a is measured by as and cos a by ac. Take a very small angle δa, and mark off from N the two angles $(a - \frac{1}{2}\delta a)$ and $(a + \frac{1}{2}\delta a)$. The horizontal and vertical projections of δa (parallel to as and ac) are evidently the increments of the sine and cosine
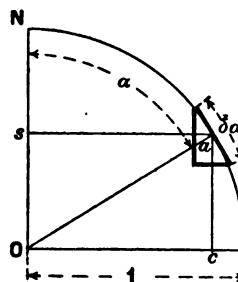
FIG. 13.

for the angle increment δa. The horizontal projection is a positive increment of the sine.; the cosine decreases as a increases, so that the vertical projection is the decrement or negative increment of the cosine. If δa be taken small enough to justify the short arc being taken as a straight line, δa and its two projections form a small right-angled triangle of the same shape as **Oas**. We have, therefore,

Increment of $\sin a = \delta(\sin a) = $ Horizontal projection of δa

$$= \frac{ac}{a0}.\delta a = \cos a \, \delta a$$

and

Decrement of $\cos a = -\delta(\cos a) = $ Vertical protection of δa

$$= \frac{as}{a0}.\delta a = \sin a \, \delta a \ .$$

Integrating these increments between any limits a_1 and a_2, the results are

$$\int_{a_1}^{a_2} \cos a \, da = \sin a_2 - \sin a_1$$

and $$\int_{a_1}^{a_2} \sin a \, da = \left[-\cos a \right]_{a_1}^{a_2} = \cos a_1 - \cos a_2 .$$

The student should carefully follow out this integration on the diagram through all four quadrants of the complete circle, paying attention to the changes of sign.

Written as indefinite integrals these results are

$$\int \cos a \, da = \sin a + C$$

and $$\int \sin a \, d = C - \cos a .^{*}$$

Expressed in words, this is, the angle-gradient of the sine of an angle is its cosine, and that of its cosine is its sine taken negatively.

75. Integration through 90°.—Since $\sin 0° = 0$ and $\cos 0° = 1$, while $\sin 90° = 1$ and $\cos 90° = 0$, we find, integrating between the limits 0° and 90°,

$$\int_{0°}^{90°} \cos a \, da = 1$$

and also $$\int_{0°}^{90°} \sin a \, da = 1 \ .$$

* See Classified List, VI. 1 and 2.

76. Spherical Surface.—Let this result be applied to the calculation of the area of the earth's surface, assuming it to be spherical. The whole surface may be divided up into narrow rings of uniform width lying between parallels of latitude. Thus, if the difference of latitude be taken to be $\frac{1}{4}°$, the uniform width of each ring will be about $17\frac{1}{4}$ miles. The meridian arc throughout this length may be considered straight without appreciable error. The ring at the equator forms practically a cylindrical ring of radius equal to that of the earth, R. A ring taken at latitude λ has a mean radius $R \cos \lambda$; and the circumferential length of its centre line is therefore $2\pi R \cos \lambda$. Naming the difference of latitude for one ring $\delta\lambda$, the width of the ring is $R.\delta\lambda$, and its area therefore $2\pi R \cos \lambda.R.\delta\lambda = 2\pi R^2 \cos \lambda.\delta\lambda$. The factor $2\pi R^2$ being the same for all the rings, we may first sum up all the products $\cos \lambda.\delta\lambda$, and afterwards multiply this sum by the common factor $2\pi R^2$. If we perform this integration from the equator to the north pole, that is, between the limits $\lambda = 0°$ and $\lambda = 90°$, we obtain the surface of the hemisphere. The integral of $\cos \lambda.\delta\lambda$ from $0°$ to $90°$ is 1; and therefore the hemispherical surface is $2\pi R^2$, and the whole spherical surface $4\pi R^2$. We used this result in § 70, p. 35.

77. Spherical Surface integrated otherwise.—The above total is $2\pi R \times 2R$. Here $2\pi R$ is the circumference of a cylinder touching the sphere, and 2R is the diameter of the sphere; so that the whole spherical surface equals that of a touching cylindrical surface whose length equals the diameter (or length) of the sphere.

In fig. 14 this circumscribing cylinder is represented by its axial section nn, ss. For each strip of spherical surface of radius r bounded by parallels of latitude $\lambda\lambda$, $\lambda\lambda$, there corresponds a strip of cylindric surface ll, ll of radius R, which latter is, in fact, the radial projection on the cylindric surface of the spherical strip. It is easy to prove that the arc $\lambda\lambda$

FIG. 14.

is greater than its projection ll in the same ratio that R is greater than r. Hence the areas of the two differential strips are equal; and, therefore, the integral areas from end to end are also equal. This proof is more elementary than that given in the previous paragraph.
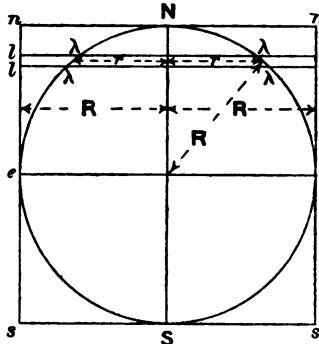
Here π is a constant multiplier, and, if any other constant multiplier k were used, the result would be

$$\frac{d}{dr}\left(kr^2 + C\right) = 2kr.$$

FIG. 10.

Written in the inverse manner, the same result appears as

$$\int 2kr\,dr = kr^2 + C.*$$

62. Rectangular Area.—A third simple example is shown in fig. 11. Here a vertical line of length k is moved horizontally to the right from an initial position x_1. Its lower end moves along the axis of x. Its upper end moves along a horizontal straight

FIG. 11.

line at the height k above this axis. The vertical sweeps out a rectangular area equal to k multiplied by the length of horizontal movement. At any stage x of the movement the area swept out

* See Classified List, III. A. 2.

is $k(x - x_1)$. Here the variable is x. Call its increment δx. While the vertical moves horizontally δx, the area swept out is a narrow rectangular strip equal to $k\delta x$. This is the increment of the integral area. Thus the x-gradient of the area is k. Calling the constant $-kx_1$ by the letter C, this result may be written in the two forms

$$\frac{d}{dx}(kx + C) = k$$

and

$$\int k\,dx = kx + C .*$$

This result is identical in form with that of § 60, fig. 9, except that in the latter the integration constant is zero and does not appear, and the variable is named p, while x is the name adopted in fig. 11. The choosing of one or other name to indicate the variable is, of course, of no consequence; but taken as two illustrations of purely geometric integration, the two results are of *different kinds*, although taken as formulas simply they are identical.

63. **Triangular Area.**—In fig. 12 we have an area swept out by a vertical line as it moves from left to right, and increases in height at a uniform rate during the motion. Its lower end lies always in the axis of x. Its upper end lies in a straight line whose inclination to the x-axis is called m.

FIG. 12.

During the movement δx, from $(x - \frac{1}{2}\delta x)$ to $(x + \frac{1}{2}\delta x)$, the mean height of the small strip of area swept out is $(k + mx)$, and its area, therefore, is

$$(k + mx)\delta x .$$

The whole area swept out during the motion from the lower limit x_1 to any upper limit x may be divided into two parts; one rectangular with the base $(x - x_1)$ and the height $(k + mx_1)$; the other triangular with the same base and the height $m(x - x_1)$.

* See Classified List, III. A. 1.

The area of the rectangle formed by the axes and the ordinates at any point is $x \times \dfrac{1}{x} = 1$; constant for all points of the curve. These two rectangles at the two points x_1 and x overlap each other, having the common area $x_1 \times \dfrac{1}{x}$ as part of each. Subtract this common part and there is left

$$x_1\left(\frac{1}{x_1} - \frac{1}{x}\right) = (x - x_1)\frac{1}{x}$$

or

$$\frac{\dfrac{1}{x_1} - \dfrac{1}{x}}{x - x_1} = \frac{1}{x_1 x}.$$

This is the ratio of the *decrease* of $\dfrac{1}{x}$ to the *increase* of x. When the increments are made minutely small, $\dfrac{1}{x_1 x}$ becomes practically $\dfrac{1}{x^2}$. In the figure a small increment of x, viz., δx, is set off equally below and above x. The above equality of areas means the equality of the two narrow strips of area rafined over in the figure. The equality is, therefore,

$$\{x - \tfrac{1}{2}\delta x\} \cdot \delta\left(\frac{1}{x}\right) = \left\{ \frac{1}{x} - \tfrac{1}{2}\delta\left(\frac{1}{x}\right) \right\} \delta x.$$

Adding $\tfrac{1}{2}\delta x . \delta\left(\dfrac{1}{x}\right)$ to each side and writing $\dfrac{d\dfrac{1}{x}}{dx}$ instead of $\dfrac{\delta\left(\dfrac{1}{x}\right)}{\delta x}$; changing also the sign, because $\dfrac{1}{x}$ decreases while x increases, we have

$$\frac{d\dfrac{1}{x}}{dx} = -\frac{1}{x^2}.$$

Expressed in words this is:—The x-gradient of the reciprocal of x is minus the reciprocal of the square of x. Writing this result inversely, we have

$$\int \frac{dx}{x^2} = C - \frac{1}{x} \; {}^*$$

* See Classified List, III. A. 2.

where C is the integration constant to be determined by special limiting conditions.

80. x-Gradient of Xx and Inverse Integration. Formula of Reduction.—In fig. 17 there is drawn a curve whose ordinates are called x and X. X represents any function of x. x is taken to the middle of δx; and, since the arc-length corresponding to δx is of minute length and may therefore be considered as straight, the point xX on the curve bisects this arc-length and also bisects δX. Also the horizontal and vertical lines through the point

FIG. 17.

xX on the curve divide the rectangular area ab into four equal parts, each $\frac{1}{4}\delta x.\delta$X.

The increase of the rectangular area Xx due to the increase δx of x is, therefore,

$$(X + \tfrac{1}{2}\delta X)(x + \tfrac{1}{2}\delta x) - (X - \tfrac{1}{2}\delta X)(x - \tfrac{1}{2}\delta x) = X\delta x + x\delta X$$

by actual multiplication, the first and fourth terms of each product cancelling out.

The first of these two terms of this increment is the strip of area between the two dotted verticals of height X; the second is the strip between the two dotted horizontals of length x. These two strips overlap each other by the $\frac{1}{4}(ab)$ small rectangle, and this has to be taken twice to obtain their sum. This compensates for the two strips not covering the outer small $\frac{1}{4}(ab)$ rectangle.

Dividing by δx, and taking minutely small increments, that of (Xx) being called $d(Xx)$, and the x-gradient of X being called X', there results

$$\frac{d(Xx)}{dx} = X + X'x.$$

According to §§ 38 and 40, pp. 16 and 17, the X-subtangent measures X'x; therefore the present x-gradient equals the sum of the function X and its X-subtangent. This X-subtangent is shown in fig. 17, where it is also graphically added to X.

The result written in the inverse integration-symbolism is

$$\int (X + X'x)dx = Xx + C.$$
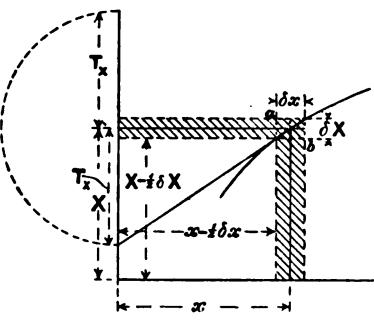
As explained below in § 83, the integral $\int(X + X'x)dx =$ $\int X dx + \int X'x dx$. Therefore the result of this article may be written

$$\int X dx = Xx - \int X'x dx + C.$$

This is an important " Reduction Formula." *

81. x-Gradient of X/x and Inverse Integration.—In fig. 18 a curve is drawn whose ordinates are called x and X, any function of x. From two points x,X and $(x+\delta x)$, $(X+\delta X)$
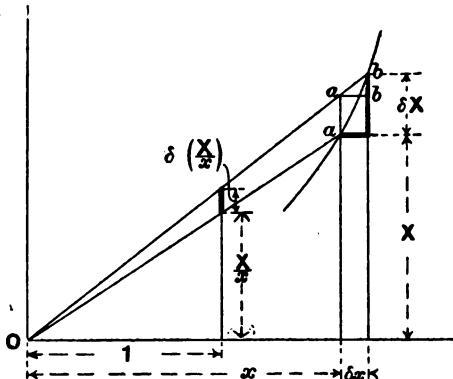
FIG. 18.

on this curve are drawn two straight lines to the origin O; and on these two lines lie the upper extremities of verticals drawn at the horizontal distance 1 from O. Evidently these last verticals measure the ratios $\dfrac{X}{x}$ and $\dfrac{X+\delta X}{x+\delta x}$. The difference between them is the increase of the ratio $\dfrac{X}{x}$ due to the increase δx of x, and is marked $\delta\left(\dfrac{X}{x}\right)$ in the figure. It is less than the small height aa in the ratio of 1 to x; and this height aa is less than δX by bb. This small height bb bears the same ratio to δx as $(X+\delta X)$ bears to $(x+\delta x)$. Thus

* See Classified List, I. 8 and II. K.

$$\delta\left(\frac{X}{x}\right)=\frac{1}{x}\left(\delta X-\frac{X+\delta X}{x+\delta x}.\delta x\right).$$

Therefore, dividing by δx,

$$\frac{d\left(\dfrac{X}{x}\right)}{dx}=\frac{X'}{x}-\frac{X}{x^2}$$

where, since extremely minute increments are taken, $\dfrac{X}{x}$ is substituted for $\dfrac{X+\delta X}{x+\delta x}$.

CHAPTER IV.

IMPORTANT GENERAL LAWS.

82. Commutative Law.—If $kX'\delta x$ is to be integrated, where to each $X'\delta x$ the same **constant** multiplier k is to be applied, it is evidently allowable to sum up first the series of products $X'\delta x$, and then to multiply this sum by k. Symbolically written this is

$$\int kX'dx = k\int X'dx$$

taken between the same limits in either case.*

Reverting to the graphic representation of integration in fig. 5, the proposition means that if there be two curves drawn, of which one has at each x its height k times the other, then the first has at each x its gradient also k times as steep as that of the other.

83. Distributive Law.—If there be two curves such as in fig. 6, the height of one being called X' and that of the other Ξ', then a third curve may be drawn, of which the height is $(X'+\Xi')$. The area under the first curve is $\int X'dx$; that under the second is $\int\Xi'dx$; that under the third is $\int(X'+\Xi')dx$. For each δx at the same x, the area of the narrow strip $(X'+\Xi')\delta x$ for the third curve equals the sum of the two strips $X'\delta x$ and $\Xi'\delta x$ for the first two

* See Classified List, I. 4.

curves. Since this is true of each strip, it is true of the whole areas; or,

$$\int X' dx + \int \Xi' dx = \int (X' + \Xi') dx \;*$$

the limits of x being taken the same in all three curves. The two curves may represent entirely different functions of x, subject only to the one condition that they must be of the same kind, it being impossible to add together quantities of different kinds. In both integration and differentiation this proposition is more frequently used by way of splitting up a whole integral into parts easier to deal with taken separately than by the converse process of combining parts into one whole. It may be extended to the more general formula

$$\int (X' + \Xi' + \mathcal{X}' + \text{etc.}) dx = \int X' dx + \int \Xi' dx + \int \mathcal{X}' dx + \text{etc.}$$

If the separate integrals on the right-hand side of the last equation be called X, Ξ, $\mathcal{X}$, etc.; then the differential view of the same proposition is that the

$$x\text{-gradient of } \{X + \Xi + \mathcal{X} + \text{etc.}\}$$

$$= \frac{d}{dx} \left\{ X + \Xi + \mathcal{X} + \text{etc.} \right\} = X' + \Xi' + \mathcal{X}' + \text{etc.}$$

$$= x\text{-grad.}X + x\text{-grad.}\Xi + x\text{-grad.}\mathcal{X} + \text{etc.}$$

Evidently the proposition of § 82 is only a special case of this in which $X' = \Xi' = \mathcal{X}'$, etc., etc.

84. **Function of a Function.**—In fig. 19 there is drawn a curve, the horizontal and vertical ordinates of which are called l and L. Thus L is a function of l, the nature of the function being graphically described by this curve. A second curve is drawn, the vertical ordinates to which are the same L's as for the first curve (plotted and measured to the same scale), and whose horizontal ordinates are called λ. λ is a function of L, the curve graphically characterising the form of the function. λ is a quantity which may possibly be of the same kind as l, and, if so, it might be plotted to the same scale. But the general case is that in which λ is not of the same kind as l, and cannot possibly, therefore, be plotted to the same scale, although in the diagram it is measured in the same direction.

Since for each given value of L the second curve gives a definite

      * See Classified List, I. 5.

corresponding value of λ, and the first curve gives a definite value of l; it follows that for each value of l there is a definite value of λ. In general there may be more than one value of λ for each l; but all the values of λ corresponding to one given value of l are definite. Thus λ is a definite function of l.

In the figure the l-gradient of L is represented graphically by a height obtained by drawing a tangent at the point lL, and plotting horizontally from this point a distance representing to the proper scale unity. This gradient is called L' in the figure.

The L-gradient of λ is similarly represented, the unit employed being measured vertically, and not being the same as that used in
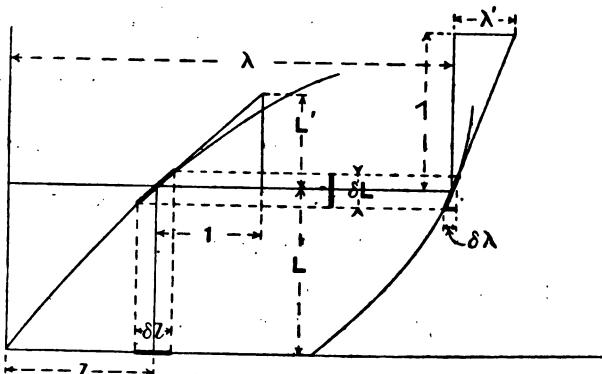
Fig. 19.

finding L', because the scales involved are different. It is marked λ'. The two gradients shown in the figure are for the same value of L; that is, the tangents are drawn at points at the same level in the two curves.

If δL and δl are the two projections of any very short length of the first curve lying partly on each side of the point where the tangent is drawn, $L' = \dfrac{\delta L}{\delta l}$. If δL and $\delta \lambda$ are the projections of any very short length of the second curve lying partly on each side of the point where the tangent is drawn, then $\lambda' = \dfrac{\delta \lambda}{\delta L}$.

If the two short arcs on the two curves be taken so as to give the same vertical projection, that is, the same δL, as is shown in fig. 19 by the dotted lines; then in the product $L'\lambda'$ the δL cancels out.

Thus,

$$\lambda' L' = \frac{\delta\lambda}{\delta L} \cdot \frac{\delta L}{\delta l} = \frac{\delta\lambda}{\delta l}$$

$$= \frac{d\lambda}{dl} \text{ when the increments are}$$

taken minutely small. In words this is :—The l-gradient of λ equals the L-gradient of λ multiplied by the l-gradient of L.

If we use the notation x, X, and F(X) instead of l, L, and λ; and if by F'(X) we understand the X-gradient of F(X) or $\frac{dF(X)}{dX}$; the same is written

$$\frac{dF(X)}{dx} = F'(X) \cdot X' \text{ or } \frac{dF(X)}{dX} \cdot \frac{dX}{dx} . *$$

85. Powers of the Variable; Powers of Sin and Cos.—This general proposition is one of the most fruitful of all laws in producing useful results when applied to particular functions, as will be seen in the next chapter.

Simple illustrations of its meaning are the following :—

Let

$$L = l^2 \text{ and } \lambda = L^3 \quad \therefore \lambda = l^6 .$$

From §§ 64 and 71 we know that

$$L' = 2l \text{ and } \lambda' = 3L^2 = 3l^4$$

$$\therefore \frac{d\lambda}{dl} = \frac{dl^6}{dl} = 6l^5 .$$

Written inversely,

$$\int l^5 dl = \frac{l^6}{6} . \dagger$$

Again, let

$$L = l^3 \text{ and } \lambda = \frac{1}{L} \quad \therefore \lambda = \frac{1}{l^3} .$$

From §§ 71 and 79,

$$L' = 3l^2 \text{ and } \lambda' = -\frac{1}{L^2} = -\frac{1}{l^6} .$$

$$\therefore \frac{d\lambda}{dl} = \frac{d\frac{1}{l^3}}{dl} = -\frac{3}{l^4} .$$

* See Classified List, I. 7 and II. 9.
† See Classified List, III. A. 2.

Written inversely,

$$\int \frac{dl}{l^4} = -\frac{1}{3l^3} . *$$

Again, let

$$L = l^3 \text{ and } \lambda = L^3 \quad \therefore \lambda = l^9$$
$$L' = 3l^2 \qquad \lambda' = 3L^2 = 3l^6$$
$$\therefore \frac{d\lambda}{dl} = \frac{dl^9}{dl} = 9l^8.$$

Written inversely,

$$\int l^8 dl = \frac{l^9}{9} . *$$

Again, let

$$L = l^2 \text{ and } \lambda = L^{4\frac{1}{2}} \quad \therefore \lambda = l^9.$$

From the last example $\frac{d\lambda}{dl} = 9l^8$, and also $L' = 2l$.   Therefore

$$9l^8 = \lambda' . 2l$$

$$\text{or } \lambda' = \frac{dL^{4\frac{1}{2}}}{dL} = 4\frac{1}{2}l^7 = 4\frac{1}{2}L^{3\frac{1}{2}}.$$

Similarly, of course, if $X = x^{4\frac{1}{2}}$, then

$$X' = 4\frac{1}{2}x^{3\frac{1}{2}}.$$

Written inversely $\int x^{3\frac{1}{2}}dx = \frac{x^{4\frac{1}{2}}}{4\frac{1}{2}}. *$

Again, let

$$L = \cos l \text{ and } \lambda = L^2 = \cos^2 l .$$

Then $L' = -\sin l \text{ and } \lambda' = 2L = 2 \cos l$

$$\therefore \frac{d \cos^2 l}{dl} = -2 \cos l \sin l = -\sin 2l .$$

Take two more examples: namely,

$$L = \sin l \text{ and } L = \cos l, \text{ while } \lambda = \frac{1}{L}.$$

Then $\lambda' = -\frac{1}{L^2}$ in both cases, and

$$L' = \cos l \text{ in } 1^{st} \text{ case}$$
$$= -\sin l \text{ in } 2^{nd} \text{ case.}$$

* See Classified List, III. A. 2.

D

Therefore, since $1/\sin l = \operatorname{cosec} l$, and $1/\cos l = \sec l$,

$$\frac{d \operatorname{cosec} l}{dl} = -\frac{\cos l}{\sin^2 l}$$

$$\text{and } \frac{d \sec l}{dl} = \frac{\sin l}{\cos^2 l}.$$

86. Reciprocal of a Function.—The second and the last of these illustrations are special cases of the semi-special semi-general case—a very important one—

$$\lambda = \frac{1}{L}.$$

Here we obtain $\dfrac{d\lambda}{dl} = \lambda' L' = -\dfrac{L'}{L^2}.$

Dividing by λ, that is, multiplying by L, we obtain this in the more symmetrical form $\dfrac{1}{\lambda} \dfrac{d\lambda}{dl} = -\dfrac{L'}{L}.$

87. Product of two Functions.—Retaining the notation of the last two articles, one particular function, to which we may apply the rule of § 84, is the product λL. Thus

$$\frac{d(\lambda L)}{dl} = \frac{d(\lambda L)}{d L} \cdot \frac{d L}{dl}.$$

But by § 80,

$$\frac{d(\lambda L)}{d L} = \lambda + L\frac{d\lambda}{d L}.$$

Multiplying by $\dfrac{d L}{dl}$, and observing that by § 84, $\dfrac{d\lambda}{d L} \cdot \dfrac{d L}{dl} = \dfrac{d\lambda}{dl}$, there results

$$\frac{d(\lambda L)}{dl} = \lambda\frac{d L}{dl} + L\,\frac{d\lambda}{dl}.$$

If X, Ξ and x be used for the three mutually dependent variables, instead of the letters λ, L and l; and if X′ and Ξ′ be the x-gradients of X and Ξ; then the above is written

$$\frac{d}{dx}(X\Xi) = X\Xi' + X'\,\Xi.$$

This extremely useful result may be easily proved directly by the

method of fig. 17, drawing a curve with ordinates X and Ⴀ, and considering the increment of the rectangular area XႠ. This increment is evidently

$$X.\delta Ⴀ + Ⴀ.\delta X$$

and dividing this by δx the above result is obtained.

Written as an integration this result is

$$\int X Ⴀ' dx + \int X' Ⴀ dx = X Ⴀ$$

$$\text{or} \quad \int X Ⴀ' dx = X Ⴀ - \int X' Ⴀ dx .^*$$

This latter form is the most fundamental and useful of the "formulas of transformation," and is usually referred to as "Integration by Parts." By its help most of the "formulas of reduction" are obtained.

88. Product of any Number of Functions.—This result may at once be extended to the product of any number of different functions X, Ⴀ, Ⴡ, etc. etc. of x. The x-gradient of this product is the sum of a number of terms, each of which is the product of all but one of the factors multiplied by the x-gradient of that one factor omitted. The result may be written in more symmetrical form if each term be divided by the product of all the factors. Thus, call the whole product X, or let

$$X = Ⴀ, Ⴡ \text{ etc., etc.}$$

then

$$\frac{X'}{X} = \frac{Ⴀ'}{Ⴀ} + \frac{Ⴡ'}{Ⴡ} + \text{etc.} + \text{etc.}$$

where X', $Ⴀ'$, $Ⴡ'$, etc., are the x-gradients of X, Ⴀ, Ⴡ, etc.

89. Reciprocal of Product of Two Functions.—The x-gradient of the reciprocal of the product of two functions Ⴀ and Ⴡ of x is found with equal ease. Call this reciprocal X and its x-gradient X'. Then

$$X = \frac{1}{Ⴀ} . \frac{1}{Ⴡ}$$

and by last paragraph, § 88 and § 86

$$\frac{X'}{X} = Ⴀ \left(-\frac{Ⴀ'}{Ⴀ^2} \right) + Ⴡ \left(-\frac{Ⴡ'}{Ⴡ^2} \right)$$

$$= -\frac{Ⴀ'}{Ⴀ} - \frac{Ⴡ'}{Ⴡ} .$$

---

* See Classified List, I. 8.

90. Reciprocal of Product of any Number of Functions.—The extension of this law to the reciprocal of the product of any number of functions is made by simply repeating the process of the last paragraph.

91. Ratio of Two Functions.—To find the x-gradient of the quotient of two functions of x, we combine the laws of § 87 and § 86. Thus, if

$$X = \frac{\Xi}{\mathfrak{X}} = \Xi \cdot \frac{1}{\mathfrak{X}}$$

then

$$\frac{X'}{X} = \frac{\Xi'}{\Xi} + \mathfrak{X}\left(-\frac{\mathfrak{X}'}{\mathfrak{X}^2}\right) = \frac{\Xi'}{\Xi} - \frac{\mathfrak{X}'}{\mathfrak{X}}.$$

It will be a useful exercise for the student to deduce this result directly by the method of fig. 18 and § 81, making the co-ordinates to the curve Ξ and $\mathfrak{X}$ instead of X and x, and without using the law $\frac{d\lambda}{dl} = \lambda'L'$.

92. Ratio of Product of any Number of Functions to Product of any Number of other Functions.—It is now apparent that all the results of §§ 86–91, inclusive, may be combined into the following single more general formula :—if

$$X = \frac{L \; M \; N, \text{ etc., etc.}}{P \; Q \; R, \text{ etc., etc.}}$$

where L, M, N, P, Q, R, etc., are functions of x, and if X', L', P', etc., be the x-gradients; then

$$\frac{X'}{X} = \frac{L'}{L} + \frac{M'}{M} + \frac{N'}{N} + \text{ etc., etc.} - \frac{P'}{P} - \frac{Q'}{Q} - \frac{R'}{R} - \text{ etc., etc.}$$

93. Theory of Resultant Error.—If the last formula be multiplied by δx, and if we call $X'\delta x \equiv \delta X$, $L'\delta x \equiv \delta L$, $P'\delta x \equiv \delta P$, etc.; all direct reference to x disappears and the formula becomes

$$\frac{\delta X}{X} = \frac{\delta L}{L} + \frac{\delta M}{M} + \frac{\delta N}{N} + \text{ etc. etc.} - \frac{\delta P}{P} - \frac{\delta Q}{Q} - \frac{\delta R}{R} - \text{ etc., etc.}$$

Now, if L, M, etc., represent measurements of physical quantities that have been made in order to calculate from them the quotient X; and if, by any means, it be known or estimated that the measurement of L has been subject to a small error δL, and that the measurement of M has been subject to a small error δM, etc., etc.; then $\frac{\delta L}{L}$ is the ratio of error in the measurement of L, and $\frac{\delta M}{M}$ is

the **ratio of error** in the measurement of M, and similarly with the other factors; while $\dfrac{\delta X}{X}$ is the **resulting ratio of error** in the **calculated** quotient X.

The above proposition may then be expressed in the following words:—

The ratio of the product of a number of measured quantities to the product of another set of measured quantities is subject to a ratio of error, equal to the sum of the ratios of error in the individual multipliers reduced by the sum of the ratios of error in the individual divisors.

Attention must here be paid to the **signs** of the errors. Thus, if δL is a negative error, then $\dfrac{\delta L}{L}$ is really a negative fraction. Similarly, if δP is a negative error, then $\left(-\dfrac{\delta P}{P} \right)$ is really a positive fraction.

Now, although there is often reason for supposing the suspected error to lie more probably in one direction than in the opposite, still errors being things which are avoided as far as possible (or convenient, or profitable), and, therefore, not consciously incurred, it is never known for *certain* whether any individual error be + or − . Thus, although divisors give by the formula ratios of error of opposite sign to those given by multipliers, it *may* happen that all the terms in $\dfrac{\delta X}{X}$ are really of the same sign, and have, therefore, all to be arithmetically added to get the whole ratio of error. Thus, if we are considering what may be the **maximum possible** error in X, we must pay no attention to the difference between multipliers and divisors, but add all the ratios of error in all the factors (both multipliers and divisors) together independently of sign.

It need hardly be pointed out that this **maximum possible** error is greater than the **probable** error.

94. Exponential Function.—In fig. 20 is drawn a curve with horizontal ordinates called l, and vertical ordinates b^l. l varies continuously. Here l is essentially a number; not, of course, necessarily a whole number, since its variation is continuous. b is a constant, and receives the name of the "**base**" of this curve. If b were any physical quantity, then its different powers would have various physical "dimensions," and would mean physical quantities of different **kinds**. In fig. 20 we assume the various vertical ordinates as all of the same kind, and therefore b cannot be a physical quantity, but must be a pure number. On this assump-

tion we find that b^l is also a pure number. In this problem, therefore, both horizontal and vertical ordinates can be nothing but pure numbers.*

At the same time it must be noted that if k be a physical constant quantity, either a unit or any other quantity, then kb^l represents a varying physical quantity of the same kind.

95. **Power Gradient of Exponential Function.**—If n be any fixed number, b^n is also a **fixed constant** number. We may write

$$b^l = b^n . b^{(l-n)}.$$

Here b^l is the vertical height of the curve, fig. 20, at any horizontal distance l; while $b^{(l-n)}$ is the height of a point on the curve at the **constant** horizontal distance n to the left of l.

Considering various l's and various pairs of points with ordinates l and $(l-n)$, we perceive that the nature of this curve gives a constant proportion, b^n, between the heights of all pairs of points at the constant horizontal distance n apart. A succession of points at the equal horizontal spacing n have their heights advancing in geometrical series, the common ratio of which is b^n.

Taking the l-gradient, or slope, of the curve at l, and remembering that b^n is a constant factor, we find

$$\frac{db^l}{dl} = b^n . \frac{db^{(l-n)}}{dl}.$$

This means that there is also the same constant proportion, b^n, between the gradients as between the heights at all pairs of points horizontally n apart.

Dividing the height by the slope we obtain the subtangent; see § 38.

$$\text{Subtangent at } l = \text{say } T = b^l \Big/ \frac{db^l}{dl}.$$

$$\text{Subtangent at } (l-n) = \text{say } T_{(l-n)} = b^{(l-n)} \Big/ \frac{db^{(l-n)}}{dl}.$$

These two last expressions are equal, because the common factor b^n cancels out in their ratio. Thus the two subtangents are equal in length; see fig. 20.

Now the length of the subtangent at l does not depend in any way upon the length n. For any one point l, we may take various lengths n, thus getting various points $(l-n)$, at **all of which** the subtangent equals that at the one point l. This means that **at all points along the curve the subtangent has the same length.**

* In Sir William Hamilton's *Quaternions*, (-1) raised to a continuously increasing power is used to indicate rotation or gradual change of direction.

Inverting the last equation,

$$\frac{db^l}{dl} = \frac{1}{T}.b^l = \text{Constant} \times b^l,$$

the constant being the reciprocal of the constant length of sub-tangent. The subtangent being a pure number, its reciprocal is also a pure number.

96. Natural, Decimal, and other Logarithms.—When $T = 1$ (unity), and the above constant is therefore also unity, the base b is designated by mathematicians by the letter e. It, e, is the base of the "**natural**" or "**hyperbolic**" or "**Neperian**" logarithms.

Each system of logarithms has a "**base**," which is a number and to which all other numbers are referred. Every number, whether whole or fractional, is represented as this "base," raised to a certain power. The power to which the base must be raised in order to produce each number is the logarithm of that number.

Thus, if b^l be a number, and if b be taken as the base of the system of logarithms, then l is the log of the number b^l to the base b.

In the "**Common**," or "**Brigg's**," or "**Decimal**" logarithms, the base is 10; every number being represented as 10 raised to one or other power.

The result of § 95 may be expressed thus:—In any system of logarithms the logarithm-gradient of the number bears a constant proportion to the number itself. The reciprocal of this constant proportion, or the subtangent in the graphic representation of fig. 20, is called the "**modulus**" of the system of logarithms.

The "natural" system of logs may be defined as that which gives the rate of increase of the number compared with its logarithm equal to the number itself; or which makes the logarithm-gradient of the number equal to the number itself; or, symbolically,

$$\frac{de^l}{dl} = e^l$$

the constant for this base e being unity. The base e is calculated to be $2\cdot71828\text{----}$. The modulus of this system is 1.

97. Number Gradient of Logarithm.—If we call the number N, then in fig. 20 the vertical height of the curve is N and the horizontal ordinate is its logarithm. The curve is sometimes termed a logarithmic curve; sometimes an exponential curve. It takes different forms according to the logarithmic base b used in drawing it out. In fig. 21 these variations of form are shown, the five curves having the bases

$$b = 2, \; e, \; 8, \; 10, \text{ and } 12.$$

All these curves come to the same height $N = 1$ at $l = 0$. At $l = 1$ the height of each is the base b. To the left of the vertical axis where l is negative, the height N is always less than unity, and decreases asymptotically to the horizontal axis towards zero height. Fig. 22 shows the extension of these same curves to high numbers with positive logarithms. The logarithm to the base e of the number N is written $\log_e N$. The logarithm to the base b of N is written $\log_b N$. Thus the decimal log of N may be unambiguously written $\log_{10} N$.

The results of §§ 95 and 96 written in this notation, and taking the reciprocal or conjugate gradient from the vertical axis, instead of the l-gradient, are

$$\frac{d \log N}{dN} = \frac{dl}{dN} = \frac{T}{N},$$

the constant T being the modulus of the system of logarithms; and

$$\frac{d \log N}{dN} = \frac{dl}{dN} = \frac{1}{N}$$

if natural logs are used in which the base is e and the modulus unity.

98. Relation between Different Log "Systems."—Any base b may be looked on as the base e raised to a certain power, which we will call $\frac{1}{T}$. Thus

$$b = e^{\frac{1}{T}} \text{ and } e = b^{T}.$$

Therefore, if

$$N = b^l = e^{\frac{l}{T}};$$

also

$$\log_b N = l = T \log_e N = \log_e N . \log_b e.$$

Thus there is a constant ratio $T = \log_b e$ between the logarithms of numbers to the base b to their logarithms in the "natural" system.

Taking the N-gradients of these last logs and remembering that T is a constant,

$$\frac{d \log_b N}{dN} = T \frac{d \log_e N}{dN} = \frac{T}{N}$$

which shows that the T of this paragraph means the same as the T of the last, § 97, or the subtangent of fig. 20.

Written as an integration result, this is

$$\int \frac{dN}{N} = \frac{\log_b N}{\log_b e} = \log_e N.^*$$

It is useful to notice that $a^{\log_b x} = x^{\log_b a}$, because taking logs on each side, we find $\log_b x . \log_b a = \log_b a . \log_b x$.

It follows that $\int a^{\log_b x} dx = \dfrac{x^{1+\log_b a}}{1+\log_b a}.\dagger$

99. Base of Natural Logs.—Although the calculation of e, or of the modulus T of any logarithmic system, is very laborious, there is no other difficulty about it beyond its tediousness. Thus to find the modulus of, say, the decimal system, an extremely small root of 10 has to be extracted. Thus the square root may be extracted, say, 100 times over. This will give $(\frac{1}{2})^{100}$th power of 10, which is a number a very minute fraction over 1. Now the decimal logarithm of this number is $(\frac{1}{2})^{100}$, which is easy to calculate, and the logarithm of 1 is zero. The former is, therefore, the increment of the logarithm corresponding to the number increment from 1 to the $(\frac{1}{2})^{100}$th power of 10. If the ratio of this logarithm increment to the number increment be multiplied by the number, which is here 1, the product is the decimal log-modulus.

The laborious part of the operation is finding the $(\frac{1}{2})^{100}$th power of 10. Extracting only a higher root, the increments will be larger, and the calculation will not give so great accuracy. Thus, if the extraction of the square root be repeated only 10 times, the result of the calculation gives

$$T_{10} = 0\cdot 434 \ 294 \ 116 \ \ddagger$$

whereas by more minute calculation its true value to 9 decimal places has been found to be

$$0\cdot 434 \ 294 \ 482.$$

The error is only $0\cdot 000 \ 000 \ 366$.

100. Logarithmic Differentiation.—Taking again the function of § 92, and taking logs, we have

$$\log X = \log L + \log M + \text{etc.} - \log P - \text{etc.}$$

---

* See Classified List, III. A. 3.   † See Classified List, IV. 7.
‡ Calculated by Mr R. F. Muirhead from the value

$$\frac{\log 10^{2^{-10}} - \log 10^{-2^{-10}}}{10^{2^{-10}} - 10^{-2^{-10}}} = \frac{10^{2^{-10}}}{2^9 \left(10^{2^{-9}} - 1\right)}.$$

and differentiating with respect to x, we have by §§ 84 and 98

$$T\frac{X'}{X} = T\left\{\frac{L'}{L} + \frac{M'}{M} + \text{etc.} - \frac{P'}{P} - \text{etc.}\right\}$$

which is the same result again as was found in § 92.

This method of differentiating products of functions is called "logarithmic differentiation."

101. **Change of the Independent Variable.**—The rule deduced in § 80 was written

$$\int X dx = Xx - \int X'x dx.$$

Now $X'\delta x$ is the same as δX; so that the integral on the right hand may be written $\int x dX$. The equation becomes then

$$\int X dx = Xx - \int x dX.$$

If $\int x dX$ is easier to find than $\int X dx$, this forms a method of facilitating the latter integration. Such a transformation is called a "change of the independent variable" or "substitution."

Fig. 23 is the graphic representation of this law. Taking the
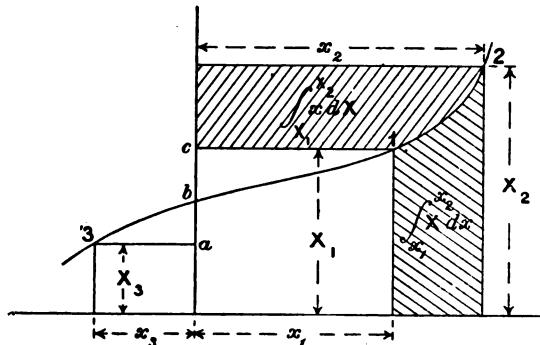
FIG. 23.

integration between the points 1 and 2 of the curve xX, the transformation is written

$$\int_{x_1}^{x_2} X dx = (X_2 x_2 - X_1 x_1) - \int_{X_1}^{X_2} x dX.$$

In fig. 23 the sum of the two areas rafined over thus /////
and \|\|\| is evidently $(X_2 x_2 - X_1 x_1)$. The first of these areas,
namely, that between the curve, the vertical axis, and the two
horizontal lines at levels X_1 and X_2, is $\int_{X_1}^{X_2} x dX$. The second, included
between the curve, the horizontal axis, and the two verticals at
distances x_1 and x_2, is $\int_{x_1}^{x_2} X dx$. Hence the above equation.

In fig. 23 the point 3 has a negative ordinate x, while its X-
ordinate is positive. A useful exercise for the student is to follow
the variations of the proposition, taking the pair of points in each
possible pair of quadrants of the full diagram.

CHAPTER V.

PARTICULAR LAWS.

102. Any Power of the Variable.—In § 99 the integral of
$\frac{1}{x}$ was found to be $\log_e x$. This is the single exception to a general
law giving the integral of any power of a variable, special examples
of which have already been demonstrated in §§ 59, 61, 71, 79,
and 86, namely, the integrals of the 0th, 1st, 2nd, $(-2)^{nd}$, 5th,
$(-4)^{th}$, 8th, and $3\frac{1}{2}^{th}$ powers.

All these examples conform to the general law

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C *$$

where C is the integration constant;

$$\text{or } \frac{dx^{n+1}}{dx} = (n+1)x^n .$$

103. Any Power of the Variable by Logarithms.—This
result can be proved to be always true by logarithmic differentia-

---

* See Classified List, III. A. 2.

tion as explained in § 100; thus, n being any power, positive or negative, integer or fractional,

$$X = x^n$$

$$\therefore \log X = n \log x$$

$$\therefore \frac{X'}{X} = \frac{n}{x}$$

$$\therefore X' = n \frac{X}{x} = nx^{n-1}$$

104. Diagram showing Integral of x^{-1} to be no Real Exception.—This formula is true for all powers of the variable, with the one exception of the integral of $\frac{1}{x}$ or x^{-1}, which is $\log_e x$.

It is often a puzzling question to students why there should be this one solitary exception to so general a law. Fig. 24 has been drawn to demonstrate graphically that it is only *formally* an exception; that it is in reality no exception at all. To compare this one apparent exception with other cases of the general law, the integrations must be taken between the same limits. It may be convenient to take the lower limit of x equal to 1 because $\log 1 = 0$ and $\left[\log x\right]_1^x = \log x$. Taking the same lower limit for $\frac{1}{n}x^n$ we have $\frac{1}{n}\left[x^n\right]_1^x = \frac{x^n - 1}{n}$ because $1^n = 1$ whatever n be.

In fig. 24 there are drawn to the scales shown, the curves $\frac{x^n - 1}{n}$ for $n = 2, 1, \frac{1}{2}, \frac{1}{10}, -\frac{1}{10}, -\frac{1}{2}, -1$, and -2. There is also plotted to the same scales the curve $\log_e x$. It will be seen that the curves for $n = \frac{1}{10}$ and $n = -\frac{1}{10}$ lie very close together, and that the curve $\log_e x$ lies between them throughout its whole length. This shows that the logarithmic curve is simply one of the general set of curves illustrating the general law, and that it is no real exception to the general law. Its position between the curves for $n = \pm\frac{1}{10}$ shows that $\log_e x$ is simply the special name given to the value of the function $\frac{x^n - 1}{n}$ when n is an excessively minute fraction, or rather when n is zero. Considering the variation of the curve in fig. 24 downwards from positive values of n to negative values of n, it is clear that the curve must have some definite position as n passes through zero, a position lying between that for small positive values of n and small negative values of n. This position is that

of the curve $\log_e x$. For $n = 0$, the function $\dfrac{x^n - 1}{n}$ takes the indeterminate form $\dfrac{x^0 - 1}{0} = \dfrac{1 - 1}{0} = \dfrac{0}{0}$, and its value has to be found by a special method, the result appearing in a special form. It should be noted that all the curves pass through the height 0 at the horizontal distance $x = 1$, and that they have here one common tangent or gradient $= 1$.

105. Any Power of Linear Function.—If a, b, and n are constants, and

$$X = (a + bx)^n,$$

we have by § 84 and last article,

$$X' = \frac{dX}{d(a + bx)} \times \frac{d(a + bx)}{dx} = n(a + bx)^{n-1} \times b = bn(a + bx)^{n-1}.$$

Written inversely for integration, this is

$$\int (a + bx)^n dx = \frac{1}{b(n + 1)}(a + bx)^{n+1} + C *$$

the constant C being introduced by the integration.

106. Reciprocal of any Power of Linear Function.—This last integration rule fails when $n = -1$.

In this case we find by §§ 51 and 98,

$$\int (a + bx)^{-1} dx = \frac{1}{b}\log_e(a + bx) + C = \frac{2 \cdot 3 \cdots}{b}\log_{10}(a + bx) + C . \dagger$$

107. Ratio of Two Linear Functions.—The function $\dfrac{ax}{b + cx}$ can be reduced so as to make it depend on the last case,

$$\text{because } \frac{ax}{b + cx} = \frac{a}{c} - \frac{\dfrac{ab}{c}}{b + cx}.$$

Therefore, by §§ 102 and 106,

$$\int \frac{ax}{b + cx} dx = \frac{ax}{c} - \frac{ab}{c^2}\log_e(b + cx) + C .$$

108. Ratio of Two Linear Functions; general case.—Since the function

$$\frac{A + Bx}{a + bx} = \frac{A}{a + bx} + \frac{Bx}{a + bx},$$

* See Classified List, III. A. 4.   † See Classified List, III. A. 5.

the integration of this function is performed by combining the results of §§ 106 and 107.

109. Quotient of Linear by Quadratic Function. — $X = a + bx^2$; then $X' = 2bx$ and $\dfrac{x}{a + bx^2} = \dfrac{1}{2b} \dfrac{X'}{X}$.

Now $\displaystyle\int \dfrac{X'}{X} dx = \int \dfrac{dX}{X} = \log_e X$ by § 98;

therefore

$$\int \frac{x}{a + bx^2} dx = \frac{1}{2b} \log_e (a + bx^2) + C$$

where C is the integration constant.

Similarly if $X = a + bx + cx^2$; then $X' = b + 2cx$, and, therefore any function of the form

$$\frac{A + Bx}{a + bx + cx^2}$$

can be readily integrated by splitting it into two terms as in § 107.*

110. Indicator Diagrams. — An important case of the use of the law of §§ 102 and 105 is the integration of the work measured by an indicator diagram.

If at any stage of the expansion p be the pressure and v be the volume of the working substance, then as the volume increases by dv, the work done is pdv.

Taking the expansion law in the more general form of § 105, or

$$p = (a + bv)^{-n};$$

then the work done during expansion from p_1, v_1 to p_2, v_2 is

$$\text{Work done W} = \int_1^2 pdv = \int_1^2 (a + bv)^{-n} dv$$

$$= \frac{1}{b(1 - n)} \left[(a + bv)^{-n+1} \right]_1^2$$

$$= \frac{1}{b(1 - n)} \left[p(a + bv) \right]_1^2.$$

Here the index is always negative. If it is arithmetically greater than 1, the expansion curve makes $\left[p(a + bv) \right]_1^2$ negative. But at the same time the divisor $(1 - n)$ is negative, so that the formula makes the work done positive. It is then better to reverse the limits and to use the positive divisor $(n - 1)$.

* See Classified List, III. A. 23.
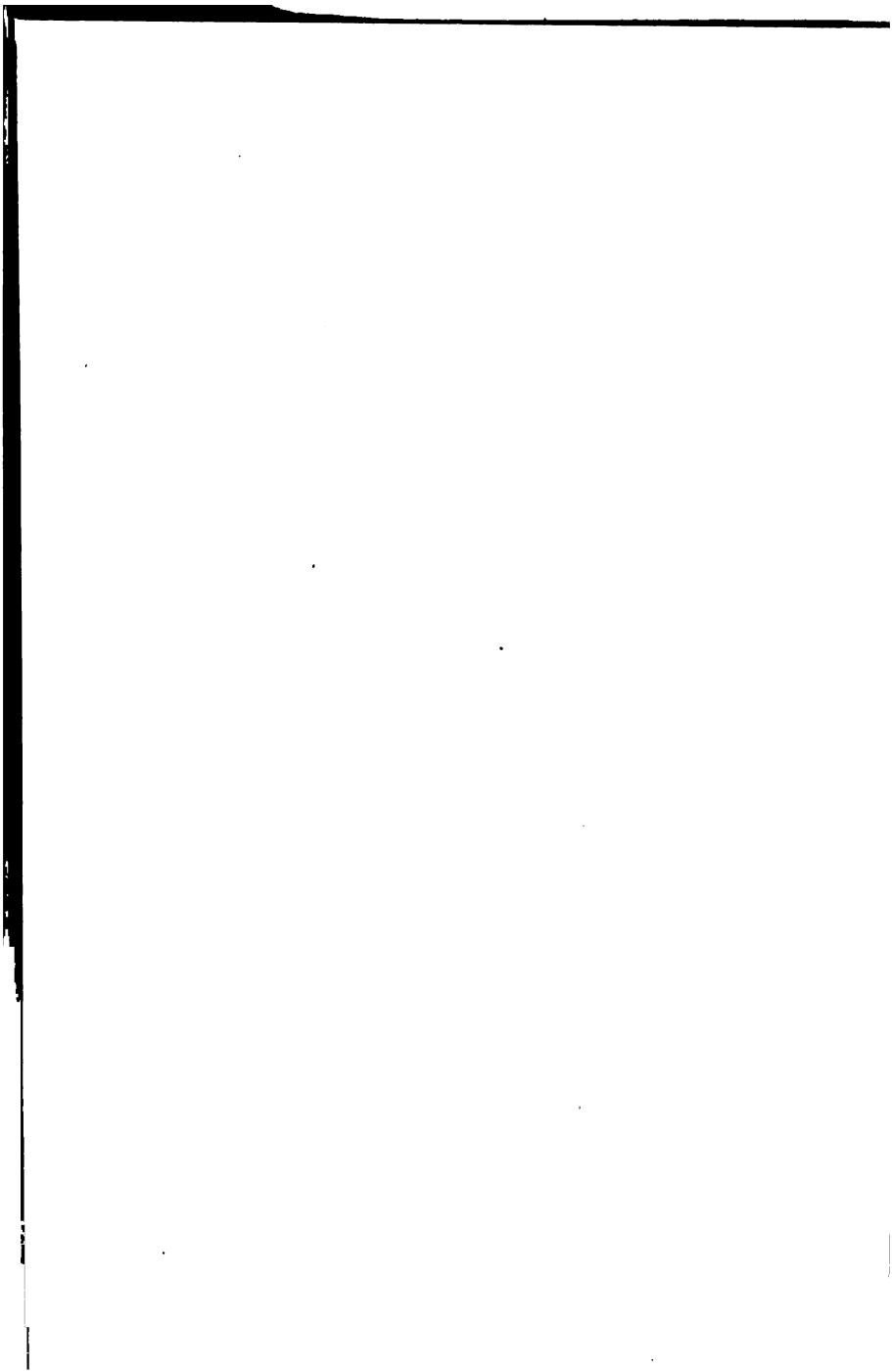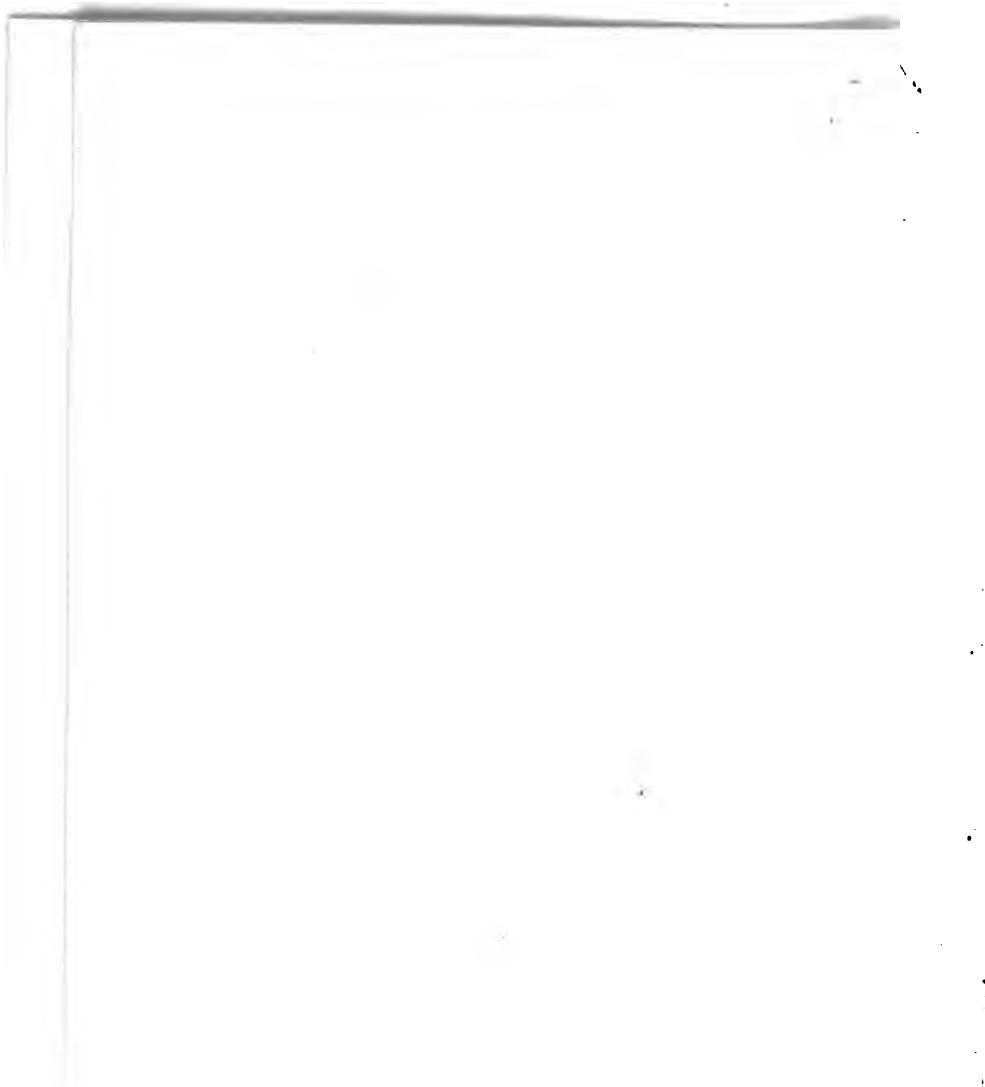
If $a = 0$, or $p = bv^{-n}$, as in most approximate formulas for expansion curves, the result simplifies, by cancelling out b from numerator and divisor, to

$$W = \frac{1}{n-1}\Big[\, pv \,\Big]_2^1$$

$$= p_1 v_1 \frac{1 - \dfrac{p_2}{p_1} \cdot \dfrac{v_2}{v_1}}{n-1}$$

$$= p_1 v_1 \frac{1 - \left(\dfrac{v_1}{v_2}\right)^{n-1}}{n-1}$$

$$= p_1 v_1 \frac{1 - \left(\dfrac{p_2}{p_1}\right)^{\frac{n-1}{n}}}{n-1}$$

$$= p_2 v_2 \frac{\left(\dfrac{v_2}{v_1}\right)^{n-1} - 1}{n-1}$$

$$= p_2 v_2 \frac{\left(\dfrac{p_1}{p_2}\right)^{\frac{n-1}{n}} - 1}{n-1} \, .$$

These formulas, which are all practically useful, give the work done during expansion in terms of the ratios between the initial and final volumes, and of the initial and final pressures; also in terms of the initial product pv and of the final product pv. The latter formula is most useful in the case of air and gas compression pumps where the initial and known volume and pressure are $v_2 p_2$.

The "**admission**" part of the indicator diagram has an area $p_1 v_1$, and this has to be added to the above, giving

$$W = p_1 v_1 \frac{n - \left(\dfrac{v_1}{v_2}\right)^{n-1}}{n-1}$$

$$= p_1 v_1 \frac{n - \left(\dfrac{p_2}{p_1}\right)^{\frac{n-1}{n}}}{n-1} \, .$$

These calculations do not take account of the back pressure deduction from the area of the card.

The "**mean pressure**," measured from zero pressure, is the last value of W divided by v_2, or

$$\frac{p_m}{p_1} = \left\{ \frac{v_1}{v_2} \cdot \frac{n - \left(\frac{v_1}{v_2}\right)^{n-1}}{n-1} \right\}.$$

From this the back pressure must be subtracted to obtain the "**effective**" mean pressure.

In the case of isothermal gas expansion, $n = 1$ or $pv = b$, and the integration for work done during expansion is

$$W = b\left[\log_e v\right]_1^2 = 2 \cdot 3 p_1 v_1 \log_{10} \frac{v_2}{v_1} = 2 \cdot 3 p_1 v_1 \log_{10} \frac{p_1}{p_2}$$

and including the work during admission

$$W = p_1 v_1 \left\{ 1 + 2 \cdot 3 \log_{10} \frac{v_2}{v_1} \right\}.$$

The ratio of mean to initial pressure is therefore

$$\frac{p_m}{p_1} = \frac{v_1}{v_2} \left\{ 1 + 2 \cdot 3 \log_{10} \frac{v_2}{v_1} \right\}.$$

111. Graphic Construction for Indicator Diagrams.—In fig. 25 the upper curve is a common hyperbola or curve of reciprocals,
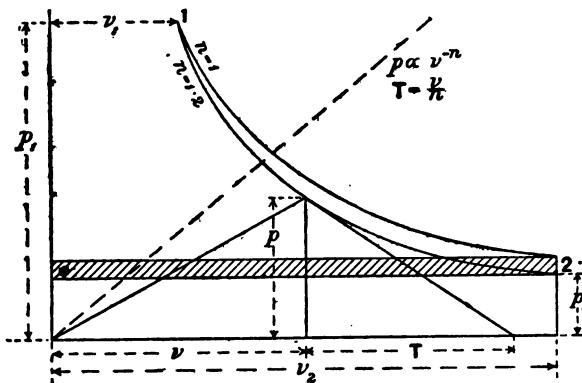

FIG. 25.

and is the gas isothermal. The lower is drawn to the formula $p = bv^{-1\cdot2}$. The product pv is the same at all points of the upper curve, and, therefore, at all points equals $p_1 v_1$. Therefore for the

point 2 on the lower curve, the horizontal strip of area rafined over equals $(p_1v_1 - p_2v_2)$; and this divided by $n - 1 = \cdot 2$, i.e., multiplied by 5, equals the work done under the lower curve during the expansion from 1 to 2.

The mean pressure, including the admission period, therefore, equals 5 times the height of the strip rafined over *plus* the height to the upper edge of the same strip.

The gradient of the curve $p = bv^{-n}$ is negative, and equals $p' = -nbv^{-n-1} = -n\dfrac{p}{v}$. Therefore $n = p' \times \dfrac{v}{p}$ omitting the *minus* sign which only indicates that the forward slope is downwards. But if T be the subtangent, then $p' = \dfrac{p}{T}$. Therefore we find $n = \dfrac{v}{T}$, and $\dfrac{1}{n-1} = \dfrac{T}{v-T}$. Thus in investigating actual indicator cards taken from engines or compressing pumps, at each point of the expansion curve at which a fair tangent can be accurately drawn, the value of the index n can be found by measuring the ratio of v to T. Also in finding the mean pressure by adding to the height of the upper edge of the rafined strip of fig. 25 the depth of this strip divided by $(n - 1)$, this division can be performed very easily by an evident graphic construction, since

$$\frac{1}{n-1} = \frac{T}{v-T}.$$

Conversely, in constructing theoretical indicator diagrams, when a few points of the curve have been calculated, it much assists in the fair drawing in of the curve to draw the tangents at these points, which can easily be done by setting off for each point $T = \dfrac{v}{n}$. If an oblique line be drawn at a tangent of inclination n to the vertical axis (it is drawn dotted in fig. 25), then at each v the height of this line will give the corresponding T. In fact, by this construction the whole curve may be accurately drawn out from point to point by drawing a connected chain of short tangents whose direction is at each point obtained in this way; the accuracy of the construction being very considerable if care be taken that each short tangent length shall stretch equally behind and in front of the point at which its direction is found by plotting T. By this construction the labour of logarithmic calculation of the heights of a series of points is rendered unnecessary.

112. **Sin^{-1}x and $(r^2 - x^2)^{-\frac{1}{2}}$.**—In § 74 it was found that the angle-gradient of a sine is the cosine, and that of the cosine *minus* the

E

sine. That is, if a be the angle and s its sine, or $\sin a = s$; then since $\cos^2 a = 1 - s^2$, we have

$$\frac{ds}{da} = (1 - s^2)^{\frac{1}{2}}.$$

When the angle is measured by its sine it is symbolically expressed as $\sin^{-1}s \equiv$ "the angle whose sine is s." Using this notation, and taking the reciprocal of the above; i.e., taking the cogradient or the "sine-gradient of the angle," we have

$$\frac{d \sin^{-1}s}{ds} = \frac{1}{(1 - s^2)^{\frac{1}{2}}}.$$

From this we deduce the more general result

$$\frac{d\left\{ a \sin^{-1}\frac{s}{r} \right\}}{ds} = \frac{a}{(r^2 - s^2)^{\frac{1}{2}}}$$

where a and r are constants.

The corresponding integrations are, when x instead of s is used to indicate the variable,

$$\int \frac{a\, dx}{(r^2 - x^2)^{\frac{1}{2}}} = a \sin^{-1}\frac{x}{r} + C$$

$$= C' - a \cos^{-1}\frac{x}{r}.^*$$

The two angles having the same fraction for sine and cosine respectively are complementary; so that these two forms of the integral only differ in the integration constants $\left(C' - C = \frac{\pi a}{2} \right)$ and in the sign of the variable parts.

The sine of an angle cannot be greater than $+1$ nor less than -1. These integration formulæ would have, therefore, no meaning in cases in which $x > r$ or $x < -r$. These limits correspond with those within which $(r^2 - x^2)^{\frac{1}{2}}$ remains real, because the square root of a negative quantity is "**impossible**." or "**imaginary**." If $(r^2 - x^2)^{\frac{1}{2}}$ arises from any actual physical problem, such a problem can never throughout the whole actual range of x make $x > r$.

118. $(1 - x^2)^{\frac{1}{2}}$ **integrated or Area of Circular Zone.**—In § 59 it was shown that the area of a sector of a circle equals $\frac{1}{2}r^2a$. The

---

* See Classified List, III. B. 6 and 5.

angle may be expressed in terms of its sine as in last article. If s be the sine, we have

$$\text{Sectorial area} = \tfrac{1}{2}r^2 \sin^{-1}s.$$

In fig. 13 this is the area NaO; in which figure the length as measures rs of the present article, and $ac = r\cos a = r\sqrt{1-s^2}$ of the present article. The triangular area aOc, therefore, equals $\tfrac{1}{2}r^2s\sqrt{1-s^2}$. Add this to the above sectorial area; the sum is the area $ONac$. This area may be taken as made up of a large number of narrow strips parallel to ON, the height of each of which would be $r\cos a = r\sqrt{1-s^2}$, while the horizontal width would be $r.ds$. The area $ONac$ is, therefore, the integral of this narrow strip of area from $a = o$ to $a = a$, which limits correspond to from $s = o$ to $s = s$.

Thus

$$\int r\sqrt{1-s^2} : rds = r^2 \int \sqrt{1-s^2}ds = \tfrac{1}{2}r^2s\sqrt{1-s^2} + \tfrac{1}{2}r^2 \sin^{-1}s$$

or

$$\int (1-s^2)^{\frac{1}{2}}ds = \frac{s(1-s^2)^{\frac{1}{2}} + \sin^{-1}s}{2}. *$$

Twice this is the area of a circular zone lying between a diameter and a parallel at the height s from the diameter, the radius being assumed 1 in the last equation.

Here, again, s cannot range outside the limits ± 1.

114. $x(r^2 - x^2)^{-\frac{1}{2}}$ **integrated.**—The function $x(r^2 - x^2)^{-\frac{1}{2}}$ may be looked on as the sine divided by the cosine, $i.e.$, the tangent of an angle, see fig. 13, while dx is the increment of the sine. The increment of the sine multiplied by the tangent evidently equals the decrement of the cosine, and accordingly the integral is *minus* the cosine, or $- (r^2 - x^2)^{\frac{1}{2}}$.†

115. $(x^2 \pm r^2)^{-\frac{1}{2}}$ **integrated.**—The function $(x^2 \pm r^2)^{-\frac{1}{2}}$ is more difficult to deal with. Let X represent any function of x, and multiply and divide its reciprocal by $(x + X)$; thus :—

$$\int \frac{dx}{X} = \int \frac{1}{X} \cdot \frac{x+X}{x+X}dx = \int \frac{1+\dfrac{x}{X}}{x+X}dx$$

$$= C + \log_e (x + X) \text{ if } X' = \frac{x}{X}.$$

The condition $X' = \dfrac{dX}{dx} = \dfrac{x}{X}$ gives

$$xdx = XdX$$

---

* See Classified List, III. B. 9.   † See Classified List, III. B. 7.

or integrating

$$x^2 + k = X^2 \text{ or } X = (x^2 + k)^{\frac{1}{2}}$$

where the integration constant k may be either $+$ or $-$. Writing $k = \pm r^2$, we have

$$\int \frac{dx}{(x^2 \pm r^2)^{\frac{1}{2}}} = C + \log_e\{x + (x^2 \pm r^2)^{\frac{1}{2}}\}.^*$$

Here, if k is negative, the differential is "imaginary" and cannot occur in any physical problem except for values of x greater than $\sqrt{-k}$.

116. $\{x(r^2 - x^2)\}^{-\frac{1}{2}}$ integrated.—The integral of $x^{-1}(r^2 - x^2)^{-\frac{1}{2}}$ is found most easily by substituting $\dfrac{1}{X}$ for x. Thus

$$X = \frac{1}{x}; \ X' = -\frac{1}{x^2} = -X^2; \ \frac{dx}{x^2} = -dX.$$

Therefore

$$\int \frac{dx}{x(r^2 - x^2)^{\frac{1}{2}}} = \frac{1}{r}\int \frac{dx}{x^2\left(\frac{1}{x^2} - \frac{1}{r^2}\right)^{\frac{1}{2}}} = -\frac{1}{r}\int \frac{dX}{\left(X^2 - \frac{1}{r^2}\right)^{\frac{1}{2}}}$$

$$= C' - \frac{1}{r}\log_e\left\{X + \left(X^2 - \frac{1}{r^2}\right)^{\frac{1}{2}}\right\} \text{ by § 113}$$

$$= C - \frac{1}{r}\log_e \frac{r + (r^2 - x^2)^{\frac{1}{2}}}{x}.\dagger$$

117. Log x integrated.—The integral of the logarithm of a variable number N is found by help of the formula of reduction in § 80 and by § 98, thus :—

$$\int \log_b N dN = N \log_b N - \log_b e \int \frac{N}{N} dN + C$$

$$= N\{\log_b N - \log_b e\} + C$$

$\log_b e$ is the "modulus" of the system of logarithms whose base is b, and for the decimal system is $0\cdot4343$ nearly. Therefore,

$$\int \log_{10} N dN = N\{\log_{10} N - \cdot4343\} + C.\ddagger$$

---

* See Classified List, III. B. 6, 3, and 4.
† See Classified List, III. B. 13 and 10.
‡ See Classified List, IV. 4.

118. Moment and Centre of Area of Circular Zone.—With the notation already used, we saw in § 113 that a narrow strip of the area of a semicircle is $2r^2(1-s^2)^{\frac{1}{2}}ds$. The distance of this strip from the diameter from which the angle and its sine are measured is rs, and the product of the area by this distance is the moment of the strip-area round this diameter. This is $2r^3s(1-s^2)^{\frac{1}{2}}ds$, in which r is a constant while s varies. Since $s = -\frac{1}{2}\dfrac{d(1-s^2)}{ds}$, the integral moment of all the strips for a zone between the diameter and a parallel rs away from it is easy to find. Calling $(1-s^2)$ by letter S, we find

$$\text{Integral moment} = 2r^3 \int s(1-s^2)^{\frac{1}{2}}ds$$

$$= -r^3 \int S^{\frac{1}{2}}dS$$

$$= -\tfrac{2}{3}r^3 S^{\frac{3}{2}}$$

$$= \tfrac{2}{3}r^3\{1 - (1-s^2)^{\frac{3}{2}}\}$$

when taken from lower limit $s = 0$ or $S = 1$.

From this is deduced by dividing by the area of the zone;

Distance of centre of area of zone from diameter

$$= \tfrac{2}{3}r\frac{1 - (1-s^2)^{\frac{3}{2}}}{s(1-s^2)^{\frac{1}{2}} + \sin^{-1}s}.$$

For the whole half circle, $\sin^{-1}s$ becomes a right angle or $\dfrac{\pi}{2}$; while $s = 1$ and $(1-s^2) = 0$. Therefore the centre of area of a semicircle is distant from the centre of the circle by

$$\tfrac{2}{3}r \cdot \frac{1}{\dfrac{\pi}{2}} = \frac{4}{3\pi}r = \cdot 4244r.$$

The moment of the whole semicircular area round the diameter is $\tfrac{2}{3}r^3$.

The integration performed here is a geometrical illustration or proof of the general integral of $x(1-x^2)^{\frac{1}{2}}$.*

119. $(r^2 + x^2)^{-1}$ integrated.—In § 78 it was found that the angle-gradient of the tangent equals the square of the reciprocal of the cosine. Remembering that

$$\cos^2 a = \frac{1}{1 + \tan^2 a}$$

* See Classified List, III. B. 9.

we find

$$da = \frac{d\tan a}{1 + \tan^2 a}.$$

Call $\tan a \equiv t$, and therefore $a = \tan^{-1} t$; we then obtain the integration

$$\int \frac{dt}{1 + t^2} = \int da = a = \tan^{-1} t + C.^*$$

Here t is essentially a number or pure ratio, and it may vary from $-\infty$ to $+\infty$. If, in order to make the formula of more general application, we introduce a constant r^2 as follows, then t may be any $+$ or $-$ physical quantity, but t and r must be of the same kind. Then, since

$$\frac{d\left(\frac{t}{r}\right)}{dt} = \frac{1}{r}$$

$$\int \frac{dt}{r^2 + t^2} = \frac{1}{r}\tan^{-1}\frac{t}{r} + C.\dagger$$

120. $(r^2 - x^2)^{-1}$ **integrated.**—Since $(r^2 - x^2) = (r + x)(r - x)$, we find easily

$$\frac{1}{r^2 - x^2} = \frac{1}{2r}\left(\frac{1}{r + x} + \frac{1}{r - x}\right)$$

and therefore this function can be integrated by help of § 106.

121. Hyperbolic Functions and their Integrals.—The functions $\frac{1}{2}(e^x + e^{-x})$ and $\frac{1}{2}(e^x - e^{-x})$ enter largely into the geometry of the hyperbola and of the catenary, as well as into the investigation of several important stress and kinetic problems. From the origin of their utilisation by mathematicians they are called **hyperbolic functions**, and from certain useful analogies between the geometry of the hyperbola and that of the circle, the names sinh x and cosh x are applied to them.$\ddagger$ In § 96 we have already found the x-gradient of e^x. Using it, we obtain

$$\frac{d\sinh x}{dx} = \frac{d}{dx}\{\tfrac{1}{2}(e^x + e^{-x})\} = \tfrac{1}{2}(e^x - e^{-x}) = \cosh x$$

and

$$\frac{d\cosh x}{dx} = \tfrac{1}{2}(e^x + e^{-x}) = \sinh x.$$

* See Classified List, III. A. 8.
+ See Classified List, III. A. 10.
‡ See Classified List, V.

Written as for integration these results are

$$\int \cosh x \, dx = \sinh x$$

and

$$\int \sinh x \, dx = \cosh x \, .$$

The integrals of other hyperbolic functions are equally easy to find, and are tabulated in Section V. of the *Classified Reference List* at the end of this book.

CHAPTER VI.

TRANSFORMATIONS AND REDUCTIONS.

122. Change of Derivative Variable.—From the fundamental idea of X′, the x-gradient of the function X, as explained in § 35, we have in equations between increments or between integrals

$$dX = X'dx$$

or

$$dx = \frac{dX}{X'} \, .$$

Now, it may be easier to find $\int \frac{X}{X'} dX$ than to find $\int X dx$, and it amounts to the same thing. The former integration may be easier if X′ be capable of convenient expression in terms of X. This transformation has been frequently employed already, as, for instance, in §§ 119, 118, 116, etc., etc.

More generally, if $f(X)$ be any function of X when X itself is a function of x, it may facilitate the integration to substitute

$$\int \frac{f(X)}{X'} dX \text{ for } \int f(X) dx ,$$

provided either that X′ cancels out of the expression $\frac{f(X)}{X'}$ or else that X′ is expressible in terms of X.

123. Substitution to clear of Roots.—A selection of such transformations is given in Section II. G of the Classified List, under the title of *Substitutions*.

Thus in II. G. 3 we get rid of roots by taking

$$X = (a + bx)^{\frac{1}{p}} \therefore x = \frac{X^p - a}{b} \therefore x^m = \frac{1}{b^m}(X^p - a)^m$$

and

$$\frac{1}{X'} = \frac{dx}{dX} = \frac{p}{b}X^{p-1}.$$

Therefore

$$\int \frac{x^m dx}{(a + bx)^{q/p}} = \frac{p}{b^{m+1}} \int \frac{(X^p - a)^m}{X^q} \cdot X^{p-1} dX$$

$$= \frac{p}{b^{m+1}} \int (X^p - a)^m X^{p-q-1} dX;$$

a form which is dealt with later on in § 125; but which can be integrated directly if m be an integer by expanding $(X^p - a)^m$ by the binomial theorem.

124. Quadratic Substitution.—Again, in II. G. 4, the function $f(ax^2 + bx + c)$ is transformed so as to get rid of the second term involving the first power of the variable. The proper substitution may be arrived at thus: put

$$X = x + \xi \qquad X' = 1 \qquad x = X - \xi.$$

Then

$$ax^2 + bx + c = aX^2 - 2a\xi X + a\xi^2 + bX - b\xi + c.$$

The two terms in the first power of X cancel if ξ be taken $\dfrac{b}{2a}$, and then the remaining three terms not involving X become

$$a\xi^2 - b\xi + c = \frac{b^2}{4a} - \frac{b^2}{2a} + c = \frac{4ac - b^2}{4a}$$

$$= \text{say } ak \text{ where } k = \frac{4ac - b^2}{4a^2}.$$

We have then the transformation

$$\int f(ax^2 + bx + c)dx = a \int f(X^2 + k)dX$$

with $X = x + \dfrac{b}{2a}$ and $k = \dfrac{4ac - b^2}{4a^2}$.

This transformation is used largely in dealing with quadratic surds.*

125. Algebraico-Trigonometric Substitution.—If in this last

---

* See Classified List, III. B. from 16 onwards.

expression k be positive so that its square root can be extracted, *i.e.*, if $4ac > b$ with a and c both positive; then the above integration of, say, $(x^2 + k)$ may be thrown into a trigonometrical form by help of the substitution

$$X = \tan^{-1}\frac{x}{k^{\frac{1}{4}}}, \text{ or } x = k^{\frac{1}{4}} \tan X \text{ and } \frac{1}{X'} = \frac{k^{\frac{1}{4}}}{\cos^2 X}.$$

In these terms

$$x^2 + k = k \tan^2 X + k = \frac{k}{\cos^2 X}$$

and therefore

$$\int f(x^2 + k)dx = k^{\frac{1}{4}} \int \frac{1}{\cos^2 X} f\left(\frac{k}{\cos^2 X}\right)dX.$$

Other similar conversions of algebraic into trigonometrical integrations are detailed in the II. G. Section of the Classified List.

126. Interchange of Two Functions.—In § 87 was established the transformation

$$\int X\Xi'dx = X\Xi - \int X'\Xi dx = X\Xi - \int \Xi dX$$

a special simple case of which, already stated in § 81, is $\Xi' = 1$ and $\Xi = x$, or

$$\int X dx = Xx - \int X'x dx = Xx - \int x dX.$$

This general formula may be useful when the function to be integrated, viz. $(X\Xi')$, is not as a whole directly integrable, but is, however, capable of being split into two factors, one of which (Ξ') has its integral (Ξ) directly recognisable.

127. Interchange of any number of Functions.—The operation may be extended to the integration of the product of any number of functions of x according to the result of § 88; but with the multiplication of the number of functions to be dealt with, there is an increase in the complexity of the conditions under which the formula may be useful, and, therefore, a decrease of the probability or frequency of such usefulness.

Transformations, according to this rule, are called **Integration by Parts.**

128. General Reduction in terms of Second Differential Coefficient.—If $f(X)$ be any function of X, and $f'(X)$ its X-gradient; then, X' being the x-gradient of X,

$$\frac{df(X)}{dx} = X'f'(X) \text{ by } § 84.$$

Also

$$\frac{d\frac{1}{X'}}{dx} = -\frac{1}{(X')^2}\frac{dX'}{dx}.$$

Here $\frac{dX'}{dx}$ is the x-gradient of X', and is called the "second differential coefficient of X with respect to x," or, more simply, the "second x-gradient of X." It is concisely written X''. Using this notation (X'') and applying § 126 we have

$$\int f'(X)dx = \frac{f(X)}{X'} + \int \frac{f(X)}{(X')^2}.X''dx.$$

Here the given function is $f'(X)$, and the supposition is that it is directly integrable with respect to X, but not so with respect to x. On this supposition the transformation will be of use if it is found that $\frac{f(X)}{(X')^2}.X''$ is directly, or more easily, integrable with respect to x.

129. General Reduction for X^r.—If $f'(X) = X^r$, then $f(X) = \frac{X^{r+1}}{r+1}$; so that in this case the above formula would be

$$\int X^r dx = \frac{X^{r+1}}{(r+1)X'} + \frac{1}{r+1} \int X^{r+1}.\frac{X''}{(X')^2}.dx.$$

In some cases this form may be preferable to $\int \frac{X^r}{X'}.dX$, which would be given by § 122.

130. General Reduction of $x^m X^r$.—If in § 126 one of the two functions whose product is to be integrated be x^m and the other X^r, where m and r are any constant indices, the transformation gives

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1} - \frac{r}{m+1} \int x^{m+1}X^{r-1}X'dx.$$

If this latter quantity be not directly integrable, it may still be capable of being further reduced by the application of other formulas of transformation already explained, so as to finally reduce it to a directly integrable form.

Such a formula is the base of certain **Formulas of Reduction.**

131. Conditions of Utility of Same.—The last formula given is capable of **repeated** application, provided that X' is proportional either to some power of x or to some power of X, the right-hand

integral then reducing to the same general form as the left-hand one. In either case, or again in the case $xX' = a + bX$, it is not difficult to prove that X must be of the form

$$X = a + bx^n.$$

If r be a positive integer, then X^r can be expanded into a finite series of powers of x, which when multiplied by x^m will give another series of powers of x, each term of which can be integrated separately; so that in this case no need of the above reduction formula will arise; although in some cases its use may shorten the work involved. But the formula is useful for repeated reductions if r is negative or fractional.

Various cases of such uses are given in Section IX. of the *Classified Reference List* at the end.

132. Reduction of $x^m(a + bx^n)^r$.

If $X = a + bx^n$, then

$$X' = nbx^{n-1} = \frac{n}{x}(X - a)$$

and § 130 gives

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1} - \frac{rn}{m+1}\int x^m X^{r-1}(X - a)dx$$

$$= \frac{x^{m+1}X^r}{m+1} - \frac{rn}{m+1}\int x^m X^r dx + \frac{rna}{m+1}\int x^m X^{r-1}dx.$$

Here we have $\int x^m X^r dx$ on each side. Bringing these two terms to one side, and dividing out by the sum of their numerical factors, viz. $\left(1 + \frac{rn}{m+1}\right) = \frac{m+1+rn}{m+1}$; we find

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1+rn} + \frac{rna}{m+1+rn}\int x^m X^{r-1}dx ;^*$$

a formula of reduction by which in the integration the power of X is reduced by 1, while that of x is left unchanged. The reduction of the power of X is compensated for by the multiplication (outside the sign of integration) by the factor a, which has the same "dimensions" as X.

This formula can be used inversely to pass from $x^m X^{r-1}$ to $x^m X^r$, that is, to increase the power of X by 1 without changing that of x.

* See Classified List, IX. A. 1.

If the other form of X', namely, nbx^{n-1}, be used in this transformation, there results

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1} - \frac{rnb}{m+1}\int x^{m+n}X^{r-1}dx;$$

a formula of reduction by which, while the power of X is decreased by 1, that of x is increased by n.

By the previous formula $\int x^{m+n}X^{r-1}dx$ may be converted into a quantity in terms of $\int x^{m+n}X^r dx$, and thus $\int x^m X^r$ reduced to an integral in which the power of x is raised by n, while that of X is left unaltered.

By similar transformations one can ring the changes among the integrals of the following set of nine functions, any one of which can be reduced to any other.

$$x^m X^{r+1} \qquad x^{m+n}X^{r+1} \qquad x^{m-n}X^{r+1}$$

$$x^m X^r \qquad x^{m+n}X^r \qquad x^{m-n}X^r$$

$$x^m X^{r-1} \qquad x^{m+n}X^{r-1} \qquad x^{m-n}X^{r-1}$$

The complete set of reduction formulæ for this purpose are given in Section IX. of the *Classified Reference List* at the end.

133. Reduction of rth Power of Series of any Powers of x.— Similarly if

$$X = ax^a + bx^\beta + gx^\gamma + k;$$

then § 130 gives

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1} - \frac{r}{m+1}\int x^m(\alpha ax^a + \beta bx^\beta + \gamma gx^\gamma)X^{r-1}dx.$$

The last integral may be taken in terms each of the form $\int x^p X^{r-1}dx$, and on account of the reduction from r to $(r-1)$ in the index of X, these may be more amenable to simple integration than the original $\int x^m X^r dx$.

Evidently this formula applies to the sum of any number of terms in different powers of x.

134. Special Case.—If in the last article one index $= 0$ and another $= 1$, we have

$$X = a + bx^n + cx$$

in which case a simple reduction, like that of § 132, will show that

$$\int x^m X^r dx = \frac{x^{m+1}X^r}{m+1+nr} + \frac{r}{m+1+nr}\int x^m \{(n-1)cx + na\}X^{r-1}dx .$$

135. Trigonometrical Reductions.—If in the general formula of § 126 the product $X\Xi'$ be equal to $\sin^n x$, then we may split this into the two factors $\sin x$ and $\sin^{n-1}x$, thus :—

$$X = \sin^{n-1}x \qquad X' = (n-1)\sin^{n-2}x \cos x$$
$$\Xi' = \sin x \qquad \Xi = -\cos x$$
$$X'\Xi = -(n-1)\sin^{n-2}x \cos^2 x = -(n-1)\sin^{n-2}x(1-\sin^2 x)$$
$$= (n-1)\sin^n x - (n-1)\sin^{n-2}x .$$

Therefore

$$\int \sin^n x dx = -\sin^{n-1}x \cos x - (n-1)\int \sin^n x dx + (n-1)\int \sin^{n-2}dx$$

$$= -\frac{\sin^{n-1}x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}dx *$$

by adding together on the left-hand side the two terms in $\int \sin^n x dx$ and dividing by the sum of their coefficients, namely, $1 + (n-1) = n$.

136. Trigonometrical Reductions.—Since by § 78 the angle-gradient of the tangent is the reciprocal of the square of the cosine, and since $dx = \dfrac{dX}{X'}$, we have

$$dx = \cos^2 x d(\tan x) = (1 - \sin^2 x)d(\tan x)$$

and, therefore,

$$\int \tan^{n-2}x dx = \int \tan^{n-2}x(1 - \sin^2 x)d(\tan x) = \frac{\tan^{n-1}x}{n-1} - \int \tan^n x dx .$$

Or, rearranging,

$$\int \tan^n x dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x dx .$$

By either the process adopted in this article or that of last article, the various trigonometrical formulæ of reduction are established which are set forth in order in Section IX. B. 1 to 12 of the *Classified Reference List* at the end of this book. In each case the reduction changes the index by 2, which change results from the substitution $\sin^2 x = 1 - \cos^2 x$.

* See Classified List, IX. B. 1.

If n the index to be reduced be an even integer, a repetition of the reduction will finally bring down the integral to the form $\int dx$, whose integral is x. If n be odd, the finally reduced integral will be either $\int \sin x\, dx$ or $\int \cos x\, dx$ or $\int \tan x\, dx$, etc., etc., all of which have already been demonstrated.

137. Trigonometrico-Algebraic Substitution. — If in these trigonometrical reductions n is fractional, then since $dx = \dfrac{d\mathrm{X}}{\mathrm{X}'}$ and $\dfrac{d \sin x}{dx} = \cos x$, etc., etc., we may convert the formula into that of § 132 and IX. A. 1 of the *Classified List*, thus :—

$$\int \sin^n x\, dx = \int \frac{\sin^n x}{\cos x}\, d(\sin x) = \int \frac{\sin^n x}{(1 - \sin^2 x)^{\frac{1}{2}}}\, d(\sin x)$$

$$= \int \mathrm{X}^n (1 - \mathrm{X}^2)^{-\frac{1}{2}} d\mathrm{X}$$

in which X stands for $\sin x$.

Similarly

$$\int \sin^n x \cos^m x\, dx = \int \frac{\sin^n x (1 - \sin^2 x)^{\frac{m}{2}}}{(1 - \sin^2 x)^{\frac{1}{2}}} \cdot d(\sin x)$$

$$= \int \mathrm{X}^n (1 - \mathrm{X}^2)^{\frac{m-1}{2}}\, d\mathrm{X}.$$

138. Composite Trigonometrical Reduction.—From the elementary formulæ for the sine and cosine of the sum of two angles, in terms of the sines and cosines of these angles, we may write: Since $p = (p - 1) + 1$

$$\sin px = \sin (p - 1)\, x \cdot \cos x + \cos (p - 1)\, x \sin x.$$
$$\cos px = \cos (p - 1)\, x \cdot \cos x - \sin (p - 1)\, x \cdot \sin x.$$

Therefore integrals of $(\sin^{\pm n} x \cdot \sin px)$, and $(\sin^{\pm n} x \cdot \cos px)$, and $(\cos^{\pm n} x \cdot \sin px)$ and $(\cos^{\pm n} x \cdot \cos px)$ may, by repeated application of the above formulæ, be reduced to series of integrals of the form $(\sin^{\pm n} x \cos^{m \pm} x)$, provided p be an integer. For these latter forms, see *Classified Reference List*, IX. B. 7–10.

CHAPTER VII.

SUCCESSIVE DIFFERENTIATION AND MULTIPLE INTEGRATION.

139. The Second x-Gradient.—In fig. 6 we have the area underneath the curve measured between two vertical ordinates called X, and the x-gradient of X, or X′, is the vertical ordinate to the curve. The slope of this curve at any point is thus the x-gradient of X′, or the x-gradient of the x-gradient of X. This is called the "**Second Differential Coefficient of X with respect to x**," or the "**Second x-gradient of X.**" It is written either

$$\frac{d^2X}{dx^2} \text{ or } X''.$$

In fig. 5 the vertical ordinate is X, and the gradient of the curve is therefore X′. Therefore, in this graphic representation the second x-gradient of X is the rate at which the gradient of the curve changes with advance along the x-axis. If the gradient of the curve increases with x, the diagram line curves upwards and X″ is positive; if the gradient decreases in the same direction there is downward curvature, and X″ is negative. The possible variations of sign of X″ are shown on fig. 3. If there be no curvature, i.e., if the diagram line be straight, then X″ is zero. Although X″ is greater the sharper the curvature, still X″ is not the measure of the curvature. The relation between X″ and the curvature is explained in § 145 below.

140. Increment of Gradient.—In fig. 5 if the gradient increase in the horizontal length δx by $(X'_2 - X'_1) = \delta X'$, then this divided by δx is the average rate of change of X′ throughout the length δx. If the two points be close together, then this is the actual rate of change of X′ between these two contiguous points; that is, it is the value of X″ at this part of the curve. That is, when δx is small,

$$\frac{\delta X'}{\delta x} = X'' \text{ or } \delta X' = X'' \delta x.$$

141. Second Increment.—If δx be small, then from the point $(x_1 - \tfrac{1}{2}\delta x)$ to the point $(x_1 + \tfrac{1}{2}\delta x)$, the curve of fig. 5 rises a height $X'_1 \delta x$, and from the point $(x_1 + \tfrac{1}{2}\delta x) = (x_2 - \tfrac{1}{2}\delta x)$ to the point $(x_2 + \tfrac{1}{2}\delta x)$, the rise of the curve is

$$X'_2 \delta x = (X'_1 + \delta X')\delta x = (X'_1 + X''\delta x)\delta x.$$

Thus while the rise is $X'_1 \delta x$ through the small stretch δx lying

equally on either side of the point 1, it is greater than this by $X''(\delta x)^2$ through the next equal small stretch δx lying equally on either side of the point 2. The horizontal distance between these two points is δx. Thus an advance of δx has resulted in an **increase of rise per** δx equal to $X''(\delta x)^2$. The excess of the second of these rises over the first is called the **second increment** or **second difference** of X. It might be written $\delta(\delta X)$, but the usual and neater symbol is $\delta^2 X$. Dividing by $(\delta x)^2$ we have

$$\frac{\delta^2 X}{(\delta x)^2} = X'' = \frac{\delta X'}{\delta x}.$$

Stated in words this reads :—

The second x-gradient of X equals the second increment of X per δx per δx divided by the square of δx.

It must be remembered that this equation has been derived only on the supposition that δx is taken very small. The other written symbol for X'', namely, $\frac{d^2 X}{dx^2}$ resembles in form $\frac{\delta^2 X}{(\delta x)^2}$ with d substituted for δ. It is sometimes stated that dx is the symbol for δx when δx becomes "infinitely small," and that when δx becomes "infinitely small" $\frac{\delta^2 X}{(\delta x)^2}$ equals $\frac{d^2 X}{dx^2}$; but the present writer prefers to avoid the use of phraseology which suggests reference to quantities which do not exist, and reasoning about which must, therefore, be quite unmeaning.

It should be noted that $\delta^2 X$ is of the same kind and dimensions as X, and that δx is of the same kind and dimensions as x; so that X'' is of the kind and dimensions of X/x^2.

142. Integration of Second Increment.—From the equation

$$\delta^2 X = X''(\delta x)^2$$

we obtain the other

Integral of $\delta^2 X$ = Integral of a corresponding continuous series of consecutive values of $\{X''(\delta x)^2\}$.

Remembering the meaning of $\delta^2 X$, it is evident that its integral from one definite point 1 or x_1 to any other point x equals

$$\delta X \text{ at } x - \delta X \text{ at } x_1$$
$$= \text{say } \delta X - (\delta X)_1$$

these two increments being taken for equal small δx's. Both these increments are very small; their difference is minutely small, and,

therefore, this integration as it stands can be of no practical use as a final result. But let us integrate again this minutely small difference of increments by taking all the successive values of δX from x_1 up to point x and subtracting the constant $(\delta X)_1$ from each, and then let us add all the differences. The result is (by the distributive law in integration, see § 83) the same as $\int_{x_1}^{x} \delta X - \int_{x_1}^{x} (\delta X)_1$. In the latter symbol, since $(\delta X)_1$ has a constant value, we have simply to multiply this by the number of them to be added, which number is $\dfrac{x - x_1}{\delta x}$; so that

$$\int_{x_1}^{x} (\delta X)_1 = \frac{x - x_1}{\delta x} \cdot (\delta X)_1 = (x - x_1)\left(\frac{\delta X}{\delta x_1}\right) = (x - x_1)X'_1.$$

In fig. 5 X'_1 is the gradient at the point 1, and this last, therefore, equals the rise of the tangent at the point 1 in the horizontal stretch $(x - x_1)$. Also $\int_{x_1}^{x} \delta X$ is the whole rise of the curve through this same stretch, or $(X - X_1)$. Therefore, this double integration of $\delta^2 X$, the shorthand symbol for which is $\int\int d^2 X$, gives the result

$$\int_{x_1}^{x} \int_{x_1}^{x} d^2 X = (X - X_1) - (x - x_1)X'_1$$
$$= X - xX'_1 - (X_1 - x_1 X'_1)$$

and this is evidently measured in its graphic representation (see fig. 26) by the height of the curve above its tangent at point 1.

143. Graphic Delineation.—In the process of single integration we found one constant introduced. In this double integration there are two introduced. In the above algebraic representation they are $- X'_1$ and $- (X_1 - x_1 X'_1)$.

On fig. 26 are clearly marked the graphic representations of these two constants, the first being the gradient of the curve at point 1.

The lower limit is the same in both steps of this double integration, namely, the point 1. It is not necessarily so. In the first step, the lower limit may be the point 1, and this will give the gradient-constant X'_1. In the second integration the lower limit might be taken at some other point 2, and this would give

$$\int_2 \int_1 d^2 X = (X - X_2) - (x - x_2)X'_1$$
$$= X - xX'_1 - (X_2 - x_2 X'_1)$$

leaving the one constant still *minus* the curve-gradient at 1, but making now the second constant *minus* the intercept upon the

F

vertical axis cut off by a line through 2 parallel to the tangent at 1. Otherwise, the integral is now the height of the curve above the line drawn through 2 parallel to the tangent at 1.
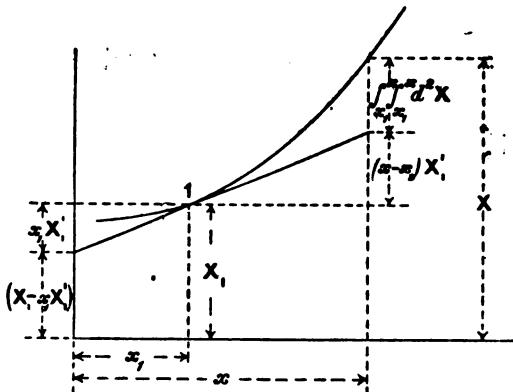

FIG. 26.

144. Integration of Second x-gradient.—Evidently $\int X''dx = X' + C$. Integrating a second time

$$\int \left\{ \int X''dx \right\} dx = \int \left\{ X' + C \right\} dx = X + Cx + K.$$

This formula is identical with the above if

$$C = -X'_1 \text{ and } K = -(X_2 - x_2 X'_1).$$

This double integration of X'' is written shorthand $\int\int X''dx\,dx$, or still more shortly $\int^{II} X''dx^2$.

We thus see that when proper attention is paid to the equality of the constants of integration,

$$\int\int d^2X = \int\int X''dx\,dx$$

or

$$\int^{II} d^2X = \int^{II} X''dx^2.$$

145. Curvature.—To show the relation between X'' and the curvature of the graphic representation, let a be the angle which the curve at any point makes with the axis of x, and let this

increase to $(a + \delta a)$ in the arc-length δa, whose horizontal projection is δx. The curvature is the reciprocal of the radius of curvature, or the "**arc-gradient of the angle,**" *i.e.*, $\dfrac{da}{da}$. Since δa is small, it equals $\tan(\delta a)$, and $\tan(\delta a)$ can be calculated by the ordinary trigonometrical rule from X' and $(X' + \delta X')$ the tangents of a and $(a + \delta a)$; thus,

$$\delta a = \tan(\delta a) = \frac{(X' + \delta X') - X'}{1 + (X' + \delta X')X'} = \frac{\delta X'}{1 + X'^2} \text{ nearly, when } \delta x \text{ is small.}$$

Also the arc-length is

$$\delta a = \sqrt{\delta x^2 + \delta X^2} = \delta x \sqrt{1 + X'^2}.$$

Therefore the curvature is, ρ being the radius of curvature,

$$\frac{1}{\rho} = \frac{da}{da} = \frac{\delta X'}{\delta x} \cdot \frac{1}{\{1 + X'^2\}^{\frac{3}{2}}} = \frac{X''}{\{1 + X'^2\}^{\frac{3}{2}}}.$$

If the radius of curvature be easily found by any direct process, the inverse form of the above relation may be useful; namely,

$$X'' = \frac{1}{\rho}\{1 + X'^2\}^{\frac{3}{2}}.$$

If T be the sub-tangent on the x-axis, and if (see fig. 27) the
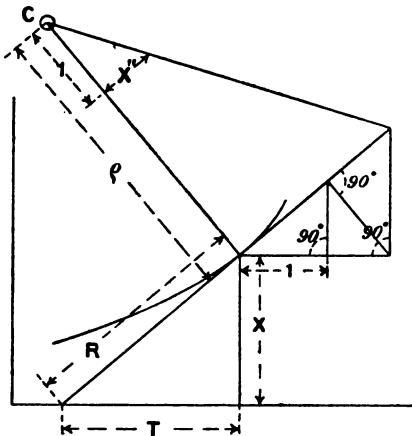
FIG. 27.

intercept on the tangent between this axis and the touching point be called R; then since $R^2 = X^2 + T^2$ and $X' = \dfrac{X}{T}$, therefore

$$1 + X'^2 = \frac{R^2}{T^2}$$

and

$$X'' = \frac{1}{\rho}\left(\frac{R}{T}\right)^3.$$

If the radius of curvature be known—(a practised draughtsman can always find it with the greatest accuracy in two or three seconds by one or two trials with the dividers)—the construction shown in fig. 27 affords a very easy graphic method of finding X'', measured to scale, according to the above formula $\frac{1}{\rho}\left(\frac{R}{T}\right)^3$. Those acquainted with the elements of Graphic Calculation will readily follow the construction from the marking of the figure without further explanation.

146. Harmonic Function of Sines and Cosines.—As an illustration of these ideas, take an ordinary harmonic curve. Let h be the height of the curve at horizontal ordinate l; let r_1, r_2 and m be constants; and let

$$h = r_1 \sin ml + r_2 \cos ml$$

then

$$h' = mr_1 \cos ml - mr_2 \sin ml$$

and

$$h'' = -m^2 \left(r_1 \sin ml + r_2 \cos ml\right) = -m^2 h.$$

The student should write out these results for the three simplified cases—(1) $r_1 = r_2 = r$; (2) $r_2 = 0$; and (3) $r_1 = 0$. In all cases he will find that $h'' = -m^2 h$.

147. Deflection of a beam.—If a beam be uniformly loaded with a load w per ft. run and have a vertical supporting force R applied at one end, the bending moment on the section distant l from this end of the beam is

$$M = Rl - \tfrac{1}{2}wl^2.$$

The bending moment diagram (ordinates M and l) is therefore a parabola. At any point l the gradient of the curve is

$$M' = R - wl$$
$$= R \text{ at the end where R acts}$$
$$= 0 \text{ at section where } wl \text{ equals R}$$

between which points it varies uniformly.

The second l-gradient of M is
$$M'' = -w$$
and is thus constant.

It is easily shown that, in a beam subjected to elastic bending only, the curvature of the (originally straight) axis equals $\dfrac{M}{EI}$, where I is the "area-moment of inertia" of the section, and E is the modulus of elasticity.

In the case of beams so stiff that the bending under safe loads is very small—which is the only case of practical interest to engineers—it is sufficiently accurate to take the curvature as the second l-gradient of the deflection, neglecting the division by the $\frac{3}{2}$ power of 1 *plus* the square of the first l-gradient.

Thus if Δ be the deflection perpendicular to l, then the second l-gradient of Δ or
$$\Delta'' = \frac{M}{EI} = \frac{1}{EI}\,(Rl - \tfrac{1}{2}wl^2) \text{ in above case}$$
and

$$\Delta = \int^{II} \Delta''\,dl^2 = \frac{1}{EI}\int^{II}(Rl - \tfrac{1}{2}wl^2)\,dl^2 \qquad \begin{array}{l}\text{if both E and I are}\\ \text{constant along } l :\end{array}$$

$$= \frac{1}{EI}\int\left(EI\,\Delta'_1 + \frac{R}{2}l^2 - \frac{w}{6}l^3\right)dl \qquad \begin{array}{l}\text{if } \Delta'_1 \text{ defines the gradient of}\\ \text{the line from which the de-}\\ \text{flection is to be measured :}\end{array}$$

$$= \Delta_2 + \Delta'_1 l + \left(\frac{R}{6}l^3 - \frac{w}{24}l^4\right)\frac{1}{EI} \qquad \begin{array}{l}\text{where } \Delta_2 \text{ defines the posi-}\\ \text{tion of the line from which}\\ \text{the deflection is measured.}\end{array}$$

At any point l the gradient is
$$\Delta' = \Delta'_1 + \left(\frac{R}{2}l^2 - \frac{w}{6}l^3\right)\frac{1}{EI}\,.$$

If the deflection be measured from a line parallel to the bent axis at its point where $wl = R$, then Δ' must equal 0 at this point, and this gives $\Delta'_1 = -\dfrac{R^3}{3EIw^2}$. If, further, we measure from a line drawn through the ends of the axis so as to make the end deflections zero, then Δ must be zero at $l = 0$, which gives $\Delta_2 = 0$.

Inserting these values of the two constants, we find
$$\Delta = \frac{1}{3EI}\left(-\frac{R^3}{w^2}l + \frac{R}{2}l^3 - \frac{w}{8}l^4\right).$$

This equals zero when $l = 0$; and, when $wl = \text{R}$, which occurs at the centre of a uniformly loaded beam freely supported at both ends, it equals

$$\Delta_c = \frac{\text{R}^4}{3\text{EI}w^3}\left(-1 + \tfrac{1}{2} - \tfrac{1}{8}\right) = -\frac{5}{24}\frac{\text{R}^4}{\text{EI}w^3}.$$

If the span be L, the whole load $= w\text{L} = \text{W}$ and $\text{R} = \tfrac{1}{2}w\text{L}$, and

$$\Delta_c = -\frac{5}{384}\frac{w\text{L}^4}{\text{EI}} = -\frac{5}{384}\frac{\text{WL}^3}{\text{EI}}.$$

148. Double Integration of Sine and Cosine Function.— In § 146 we find

$$h'' = -m^2 h$$

and, comparing this with the original equation, we see that the general result of a double integration from this relation is

$$h = r_1 \sin ml + r_2 \cos ml$$

where r_1 and r_2 are the two constants introduced by integration.

But if the second-gradient equation be of the other form

$$h'' = -m^2(r_1 \sin ml + r_2 \cos ml)$$

the result of a double integration is not the same: it is more general, namely,

$$h = r_1 \sin ml + r_2 \cos ml + \text{C}_1 l + \text{C}_2$$

where C_1 and C_2 are the two integration constants.

The former ($h'' = -m^2 h$) is a special case of the latter more general rule, in which special case $\text{C}_1 = 0 = \text{C}_2$; and this specialty gives rise to the relation $h'' = -m^2 h$, which relation does not hold good in general.

In the general formula the constant C_1 gives a choice of gradient of the line from which h is to be measured; while C_2 gives a further choice of level at which to draw this datum line. In the special case this level must be such as to make $h = r_2$ when $l = 0$ and the gradient of the datum line must be zero.

149. Exponential Function.— If $\text{X} = b^z$, then by § 95

$$\text{X}' = \frac{b^z}{\text{T}} = \frac{\text{X}}{\text{T}} \text{ and therefore } \text{X}'' = \frac{\text{X}}{\text{T}^2}.$$

If $\text{X} = b^{mz}$ where m is any constant, either positive or negative, whole or fractional; then

$$\text{X}' = \frac{m}{\text{T}}\text{X} \text{ and } \text{X}'' = \left(\frac{m}{\text{T}}\right)^2\text{X}.$$

This case is the counterpart of that in the last article where $(-m^2)$ was essentially and necessarily negative, while here $\left(\dfrac{m}{T}\right)^2$ is essentially positive.

150. Product and Quotient of two or more x-Functions.—If L and M are functions of x, and if $X = LM$, the first and second x-gradients of X are

$$X' = L'M + LM' \text{ and}$$

$$\therefore X'' = L''M + 2L'M' + M''.$$

In the case of $M = x$, then $M' = 1$ and $M'' = 0$;

$$\text{therefore } X'' = L''x + 2L'.$$

Similarly, if X be the product of three x-functions, L, M, N, then X'' is the sum of a series of terms each of which contains the three letters L M and N, and in each term the number of dashes indicating the number of differentiations will be 2. Dividing by $X = LMN$, the result may be written

$$\frac{X''}{X} = \frac{L''}{L} + \frac{M''}{M} + \frac{N''}{N} + 2\left(\frac{L'M'}{LM} + \frac{L'N'}{LN} + \frac{M'N'}{MN}\right)$$

a form analogous to that of § 92; and which may be extended to the product of any number of factors.

151. Third and Lower x-Gradients and Increments.—The x-gradient of the second x-gradient of X is the third x-gradient, and the x-gradient of this again is the fourth x-gradient; and so on through any number of differentiations.

These successive gradients are written either X''', X^{iv}, X^v, or else $\dfrac{d^3X}{dx^3}$, etc.

Similarly, if two successive values of the second increment of X per δx per δx be taken at two places δx apart, *the difference between them is called the third increment of X per δx per δx per δx.* This is written $\delta^3 X$; and if it be divided by the cube of δx, it is easy to show that $\dfrac{\delta^3 X}{(\delta x)^3} = X''' = \dfrac{d^3X}{dx^3}$ when δx is very small. This is **not a truism.** The symbol $\dfrac{d^3X}{dx^3}$ ought **not** to be considered capable of being split into two parts, one of which, the numerator, d^3X, is the value of $\delta^3 X$ when δx is very small. Nevertheless, it is correct to write for **any** very small δx

$$\delta^3 X = X'''(\delta x)^3 = \frac{d^3X}{dx^3}(\delta x)^3.$$

Again, if $\delta^n X$ be the nth increment, and $\dfrac{d^n X}{dx^n}$ the nth gradient, then for any very small δx it is correct to write

$$\delta^n X = \frac{d^n X}{dx^n}(\delta x)^n.$$

152. Rational Integral x-Functions.—If $X = kx^m$, then

$$X' = kmx^{m-1}; \quad X'' = km(m-1)x^{m-2}$$

and

$$\frac{d^n X}{dx^n} = km(m-1)(m-2)\text{----}(m+1-n)x^{m-n}.$$

If m be a positive integer, the mth gradient of kx^m will be a constant; namely,

$$\frac{d^m X}{dx^m} = km(m-1)(m-2)\text{----}3 \cdot 2 \cdot 1 = km\,!$$

Thus the $(m+1)^{th}$ gradient of kx^m is zero, as is also every lower gradient, if m be a positive integer.

But if m be fractional, then the successive gradients pass into negative powers of x, so that, in this case, a lower gradient may have a very large value for very small values of x. Thus, if $X = x^{\frac{5}{2}}$; then, $X' = \dfrac{5}{2}x^{\frac{3}{2}}$, $X'' = \dfrac{15}{4}x^{\frac{1}{2}}$, and $X''' = \dfrac{15}{8x^{\frac{1}{2}}}$; giving very large values of X''' for very small values of x.

If $X = ax^m + bx^{m-1} + \text{----} + kx$, and if m be an integer, then at each successive differentiation one term disappears, and the mth gradient is again a constant, viz., $am\,!$ Thus any terms in the above function, except the first, might be omitted without altering the mth gradient. There are, therefore, $(m-1)\,!$ different functions of the above type which give the same m^{th} gradient; $(m-2)\,!$ different ones which give the $(m-1)^{th}$ gradient the same in all; and so forth, the differences corresponding with those arising from putting any except the first of the constant factors in the above general formula equal to zero.

153. Lower x-Gradient of Sine and Exponential Functions.—The successive gradients of some functions have a re-entrant or repeating character. For instance,

$$X = k_1 \sin mx + k_2 \cos mx$$

$$X'' = -m^2 X$$

$$\therefore\ X^{iv} = m^4 X$$

$$\therefore\ X^{vi} = -m^6 X, \text{ etc., etc.}$$

Again, see § 95,

$$X = b^{\beta x}$$

$$X' = \frac{\beta}{T} b^{\beta x} = \frac{\beta}{T} X$$

$$\therefore X'' = \left(\frac{\beta}{T}\right)^2 X$$

$$\therefore \frac{d^n X}{dx^n} = \left(\frac{\beta}{T}\right)^n X$$

and this is true whether β be $+$ or $-$.

154. General Multiple Integration.—If X', X'', X''', X^{iv}, etc., are the successive x-gradients of some function X, and if we start with a knowledge of the lowest of these gradients only, and wish to work upwards to a knowledge of the higher gradients and of X by repeated integration; we find

$$\int X^{iv} dx = X''' + C_3$$

where C_3 is the constant of integration. Then

$$\iint X^{iv} dx^2 = \int X''' dx + \int C_3 dx = X'' + C_3 x + C_2$$

and

$$\int^{iii} X^{iv} dx^3 = X' + \frac{C_3}{2} x^2 + C_2 x + C_1$$

and

$$\int^{iv} X^{iv} dx^4 = X + \frac{C_3}{6} x^3 + \frac{C_2}{2} x^2 + C_1 x + C_0.$$

This result might perhaps be more clearly understood when expressed as follows:—X^{iv} may be written $(X^{iv} + 0)$. Then the proposition is that the fourth integral of the known function X^{iv} is the function X whose fourth x-gradient is X^{iv}, *plus* the function $(\frac{1}{6}C_3 x^3 + \frac{1}{2}C_2 x^2 + C_1 x + C_0)$ whose fourth x-gradient is 0.

If there were n integrations, there would be $(n+1)$ terms in the result, one of which would be a constant, and $(n-1)$ of which would be multiples of the first $(n-1)$ integral powers of x. § 152 illustrates one special example of this general proposition.

The constants are to be determined from the "**limiting conditions.**" The number of limiting conditions, a knowledge of which is necessary to definitely solve the problem, is the same as the number of "**arbitrary constants**" C appearing in the general solution.

In the above case C_2 might be determined from a knowledge of one particular value of X'', and then C_1 from that of one particular gradient X', the remaining C_0 being found from one particular value of X being given.

155. General Multiple Integration.—If in § 126 we write $\int X dx$ instead of X, and therefore X instead of X', we obtain

$$\int \{\Xi' \int X dx\} dx = \Xi \int X dx - \int \Xi X dx .$$

If in this formula the x-function Ξ be x itself, so that $\Xi' = 1$, there results

$$\int^{\text{II}} X dx^2 = x \int X dx - \int x X dx$$

which enables two possibly easy single integrations to be substituted for one double integration which may be otherwise impracticably difficult.

Conversely, a given function (xX'') may be difficult to integrate once, while the part of it X'' is recognised as the second x-gradient of a known function X, and then the form

$$\int x X'' dx = x X' - \int^{\text{II}} X'' dx^2$$
$$= x X' - X$$

may be useful.

156. Reduction Formulæ.—From § 150 we have

$$\frac{d^2}{dx^2} (xX) = xX'' + 2X'$$

from which it follows that

$$xX = \int^{\text{II}} x X'' dx^2 + 2 \int^{\text{II}} X' dx^2 .$$

In this substitute X for X', and therefore X' for X'', and $\int X dx$ for X ; there results then

$$\int^{\text{II}} X dx^2 = \tfrac{1}{2} x \int X dx - \tfrac{1}{2} \int^{\text{II}} x X' dx^2$$
$$= \tfrac{1}{2} x \int X dx - \tfrac{1}{2} \int^{\text{II}} x \, dX \, dx.$$

Again, if in the same formula there be substituted X for X'', and therefore $\int X dx$ for X' and $\int^{\text{II}} X dx^2$ for X, the result appears as

$$\int^{\text{III}} X dx^3 = \tfrac{1}{2} x \int^{\text{II}} X dx^2 - \tfrac{1}{2} \int^{\text{II}} x X dx^2 .$$

157. Graphic Diagram of Double Integration.—The meaning of double integration can be very easily represented graphically.

In fig. 5 the slope of the curve is X' and the height of the curve is X, the first integral of X' by dx. Thus ($X \delta x$) or the strip of area between two contiguous verticals under the curve is the increment of the second integral of X' by dx. Thus the area under the curve included between two given limiting verticals is their second integral, or

$$\text{Area under curve} = \int^{\text{II}} X' dx^2.$$

This graphic representation will help the student to perceive clearly that this integral is not the sum of a number of terms, each of which is the square of δx multiplied by the slope X'. The square of any one δx multiplied by the coincident slope X' would be the rectangle of base δx and height δX, because $X' \delta x = \delta X$. The sum of the series of such rectangular areas stretching between given limits on the curve is not any definite area, and it can be made as small as desired by taking the δx's sufficiently small. But this small rectangular area ($\delta x . \delta X$) is easily recognised to be the *second increment* of the area under the curve. The first difference is the area of the whole vertical strip between contiguous verticals. The difference between two such successive narrow strips (each being taken the same width δx) is the above ($\delta x . \delta X$). Thus as $X' dx^2$ is this second difference which equals $X'(\delta x)^2$, there is nothing illegitimate in considering the symbol dx^2 in $\int^{\text{II}} X' dx^2$ to represent the value of $(\delta x)^2$ when δx is taken minutely small.

158. Graphic Diagram of Treble Integration.—The idea of treble integration may be similarly represented graphically.

If the various areas in fig. 5 under the curve measured from any given lower limit up to the various vertical ordinates at the successive values of x, be looked upon as projections or plan-sections of a solid, the successive sections for each x and the following $(x + \delta x)$ being raised above the paper by the heights x and $(x + \delta x)$; then this volume is the true graphic representation of $\int^{\text{III}} X' dx^3$, because the increment of this volume, or the slice of volume lying between two successive parallel sections δx apart, is the section-area at the

middle of the thickness δx multiplied by δx. This section-area we have seen in § 157 to be $\int^{\text{II}} X'dx^2$; and, therefore, the above increment of volume is $\int^{\text{II}} X'dx^3$. The integral of this is $\int^{\text{III}} X'dx^3$.

If the lower limiting vertical ordinate of the area be at $x = 0$, then two of the side surfaces of the above integral volume are **planes** normal to the paper of the diagram and passing through the axes of x and X. A third side surface, namely, that passing through the successive X edges (which are the various upper limiting ordinates in the area integrals), is also a **plane**: it passes through the X-axis and is included at 45° to the diagram paper. The fourth side surface is in general curved. These four side surfaces, three of which are flat, give to the volumetric representation of treble integration the general form of a quadrilateral pyramid. The base of this pyramid is plane parallel to the diagram paper.

As a valuable exercise, the student should endeavour to obtain a clear mental conception of the fact that $X'(\delta x)^3$, the value of which becomes $X'dx^3$ when δx is minutely small, is the third difference in the continuous increase with x of this pyramidal volume.

CHAPTER VIII.

INDEPENDENT VARIABLES.

159. Geometrical Illustration of Two Independent Variables. —Hitherto there have been considered combinations of such functions alone as are mutually dependent on each other. The functions x, X, $\mathfrak{X}$, etc., have been such that no one of them can change in size without the others concurrently changing size.

In fig. 1, § 11, we have a vertical plane section of the surface of a piece of undulating land. Suppose it to be a meridional or north and south section. On it each distance measured northwards from a given starting-point corresponds to a definite elevation of the ground. If we take other meridional sections of the same piece of country, this same northward co-ordinate will correspond with other heights in these other sections. Thus, if h be used as a general symbol to mean the height of the surface at any and every point of it, then h depends not only on the northward co-ordinate

or latitude, but also upon the westward co-ordinate or longitude. If there be freedom to move anywhere over the surface, the two co-ordinates of latitude and longitude may be varied **independently of each other**, that is, a change in one does not necessitate any change in the other.

Under such circumstances the **elevation** is said to be a **function of two independent variables.**

160. Equation between Independent Increments.—In moving from any point 1 to any other point 2, the elevation rises (or falls) from say h_1 to h_2. Let the latitudes, or northward ordinates, of the two points be n_1 and n_2, an l the westward ordinates or longitudes be w_1 and w_2. Then the same change of elevation would be effected by either of two pairs of motions; namely, first, a motion northwards $(n_2 - n_1)$ without change of longitude, followed by a motion $(w_2 - w_1)$ westwards without change of latitude; or, second, a motion $(w_2 - w_1)$ without change of n, followed by a motion $(n_2 - n_1)$ without change of w. This is true whether these motions be large or small. Suppose them to be small, and further suppose that there are no sudden breaks in the ground, that is, that the change of elevation is **continuous** or gradual over the whole surface. Call the small northward, westward, and vertical movements by the symbols

$$n_2 - n_1 = \delta n$$
$$w_2 - w_1 = \delta w$$
$$h_2 - h_1 = \delta h .$$

Then if the meridional northward slope of the ground, just north of point 1, be called $\left(\dfrac{\partial h}{\partial n}\right)_1$, the rise during the small northward movement δn from 1 will be $\left(\dfrac{\partial h}{\partial n}\right)_1.\delta n$; and if the westward slope of the parallel of latitude through 2, just east of the point 2, be called $\left(\dfrac{\partial h}{\partial w}\right)_2$, the rise during the small westward movement δw which, following the above, completes the motion to 2, will be $\left(\dfrac{\partial h}{\partial w}\right)_2.\delta w$. The sum of these two rises gives the whole of δh, or

$$\delta h = \left(\frac{\partial h}{\partial n}\right)_1.\delta n + \left(\frac{\partial h}{\partial w}\right)_2.\delta w .$$

Here the two gradients are not gradients at the same point. If fig. 28 be a plan and two elevations of the small part of the surface

considered, they are the northward and westward gradients at ν_1 and ν_2 at the middle points of 1N and N2 in the plan.

If now the passage from 1 to 2 be effected by passing through W in fig. 28, and if $\left(\dfrac{\partial h}{\partial w}\right)_1$ and $\left(\dfrac{\partial h}{\partial n}\right)_2$ be the westward and north-


WEST
ELEVATION

NORTH
ELEVATION

PLAN
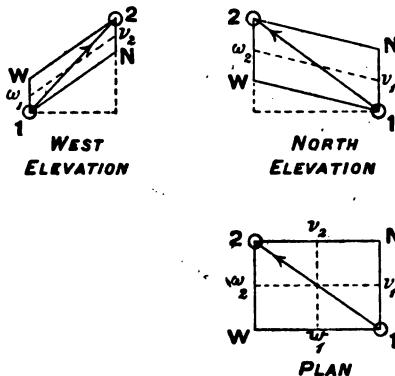
FIG. 28.

ward slopes of the ground at ω_1 and ω_2; then the same change of elevation may be calculated thus,

$$\delta h = \left(\frac{\partial h}{\partial w}\right)_1 \delta w + \left(\frac{\partial h}{\partial n}\right)_2 \delta n$$

where the δw and the δn are also the same lengths as before, the quadrilateral 1N2W being a parallelogram.

$\left(\dfrac{\partial h}{\partial w}\right)_2$ and $\left(\dfrac{\partial h}{\partial w}\right)_1$ are the westward slopes on opposite sides of this parallelogram; they are the slopes of N2 and 1W in the "North Elevation." $\left(\dfrac{\partial h}{\partial n}\right)_1$ and $\left(\dfrac{\partial h}{\partial n}\right)_2$ are the northward slopes upon the other pair of opposite sides; they are the slopes of 1N and 2W in the "West Elevation."

Adding these two equations and dividing each side by 2; and, further, calling the means between the gradients on the opposite sides of the parallelogram by the symbols $\dfrac{\partial h}{\partial n}$ and $\dfrac{\partial h}{\partial w}$; we have

$$\delta h = \frac{\partial h}{\partial n}.\delta n + \frac{\partial h}{\partial w}.\delta w .$$

On a continuous surface such as is here supposed, the above arithmetic means are, with great accuracy, equal to the actual gradients along the centre lines $\omega_1 v_2$ and $v_1 \omega_2$ of the small rectangle; that is, the gradients at the centre of the short straight line 1 2.

161. Equation between Independent Gradients.—If the short length 1 2 be called δs, so that δn and δw are the northward and westward projections or components of δs; then we have, as general truths, by dividing successively by δn, δw, and δs,

$$\frac{\delta h}{\delta n} = \left(\frac{dh}{dn}\right)_s = \frac{\partial h}{\partial n} + \frac{\partial h}{\partial w}\cdot\left(\frac{dw}{dn}\right)_s$$

$$\frac{\delta h}{\delta w} = \left(\frac{dh}{dw}\right)_s = \frac{\partial h}{\partial n}\left(\frac{dn}{dw}\right)_s + \frac{\partial h}{\partial w}$$

$$\frac{\delta h}{\delta s} = \frac{dh}{ds} = \frac{\partial h}{\partial n}\cdot\left(\frac{dn}{ds}\right)_s + \frac{\partial h}{\partial w}\cdot\left(\frac{dw}{ds}\right)_s$$

where the restrictive symbol $(\ )_s$ indicates a ratio of increments occurring concurrently along the special path s over the surface, an element of which path is 1 2 or δs; while the ratios of increments not marked with this symbol are pure northward and eastward gradients. $\dfrac{dh}{ds}$ does not need to be marked, as its terms indicate plainly that it means the actual whole gradient of the ground along the path s.

$$\left(\frac{dn}{dw}\right)_s, \left(\frac{dw}{dn}\right)_s, \left(\frac{dn}{ds}\right)_s, \text{ and } \left(\frac{dw}{ds}\right)_s$$

are different measures of the direction of the path s in plan; the first two are the tangents of the inclination of this path from the west and from the north respectively; the last two are the sines of the same inclinations. These measures of its direction particularise the special path to which the equations apply. $\dfrac{\partial h}{\partial n}$ and $\dfrac{\partial h}{\partial w}$ have no connection with, and are quite independent of, the direction of this path s: they are the due north and due west gradients at a point of the path, and depend upon the **position** of this point in the field, but not upon the direction of the path at such point.

The gradients $\dfrac{\partial h}{\partial n}$ and $\dfrac{\partial h}{\partial w}$ are called the "**partial**" differential coefficients or gradients of h with respect to n and w.

$\left(\dfrac{dh}{dn}\right)_s$ is the ratio of rise to northward progress in travelling

along the path s, and depends upon the direction of this path.
It is quite different from $\dfrac{\partial h}{\partial n}$. In fig. 28 it, $\left(\dfrac{dh}{dn}\right)_s$, equals the tangent
of inclination of the line 1 2 to the horizontal base in the "West
Elevation"; while $\dfrac{\partial h}{\partial n}$ equals the tangent of inclination of line 1N
to same base also in the "West Elevation."

Similarly, $\left(\dfrac{dh}{dw}\right)_s$ equals the tangent of inclination of 12 to the
horizontal base in the "North Elevation," while $\dfrac{\partial h}{\partial w}$ is the tangent
of inclination of line 1W to same base also in the "North
Elevation."

162. Constraining Relation between Three Variables.—We
have above considered the ordinates n and w to any point of
the surface as mutually independent of each other, and h as
dependent upon *both* n and w. But we may equally well consider
w a function dependent on *both* n and h, while looking on n and h
as mutually independent of each other. Generally between the
three functions n, w, and h there is only one restrictive rela-
tional law established, leaving one degree of freedom of variation
among the three. If a second restrictive law be imposed upon the
relations between the three, this means that we are restricted to
some particular path, such as s, over the surface, and are no
longer free to take points all over the surface.

163. Equation of Contours.—The meridional section is such
a restricted path; the restriction being $\delta w = 0$. The parallel of
latitude is another such restricted path; the restriction in this case
being $\delta n = 0$. A level contour line is a third example of such a
restricted path, the restriction being $\delta h = 0$. Therefore, if the
path s be a contour line, we have $\dfrac{dh}{ds} = 0$, and thus one form of
the equation giving the shape of a contour is

$$\frac{\partial h}{\partial n}\left(\frac{dn}{ds}\right)_s + \frac{\partial h}{\partial w}\left(\frac{dw}{ds}\right)_s = 0$$

or

$$\frac{\left(\dfrac{dn}{ds}\right)_s}{\left(\dfrac{dw}{ds}\right)_s} = \left(\frac{dn}{dw}\right)_s = -\frac{\dfrac{\partial h}{\partial w}}{\dfrac{\partial h}{\partial n}}.$$

Here $\left(\dfrac{dn}{dw}\right)_s$ = tangent of northward bearing of contour from due

west, and this is seen to equal *minus* the ratio of the due west slope to the due north slope. The *minus* simply means that if both these slopes are positive upward gradients, then the bearing is south, not north, of due west. The steeper the west slope is in comparison with the north slope, the more does the contour veer round to the south.

The geometrical linear ordinates of the above illustration may be taken as the graphic scaled representatives of any kinds of measurable quantities related to each other in a similar manner.

164. General x, y, $F(x,y)$ Nomenclature.—Let the two independent variables be called x and y, and let the function dependent on these be called $F(x,y)$. Let also the rate of change of $F(x,y)$ with x when y is kept constant be called $F'_x(xy)$, and its rate of change with y when x is kept constant be called $F'_y(x,y)$.

Then the equations of § 161 are written

$$\{F'_x(x,y)\} = F'_x(x,y) + F'_y(xy)\frac{dy}{dx}$$

$$\{F'_y(x,y)\} = F'_x(x,y)\frac{dx}{dy} + F'_y(x,y)$$

where $\{F'_x(xy)\}$ is the rate of change of $F(xy)$ with change of x when the change of x is associated with a change of y in the ratio indicated by $\frac{dy}{dx}$; this ratio $\frac{dy}{dx}$ being any whatever, but the ratio inserted on the right hand being always the same as that involved implicitly on the left hand.

165. Two Functions of Two Independent Variables.—Again, if $f(xy)$ be another similar function of x and y, then

$$\left\{\frac{dF(xy)}{df(xy)}\right\} = \frac{F'_x(x,y) + F'_y(x,y) \cdot \dfrac{dy}{dx}}{f'_x(x,y) + f'_y(x,y) \cdot \dfrac{dy}{dx}}$$

the brackets { } on the left meaning that the equation gives a particular value of the $f(xy)$ – gradient of $F(xy)$; namely, that particular value obtaining along with change of y combined with change of x in the ratio $\frac{dy}{dx}$ inserted on the right hand.

166. Applications to p, v, t and ϕ Thermal Functions.— An important example of the kind of relation described is that of temperature, pressure, and specific volume of any one definite substance. If t, p, and v indicate these, and if t'_p indicate the

G

pressure-gradient of the temperature with volume kept constant, while t'_v indicates the volume-gradient of the temperature at constant pressure; then for any changes δp and δv of the pressure and volume, there results a temperature increment

$$\delta t = t'_p.\delta p + t'_v.\delta v.$$

For any change of thermal condition in which the volume-gradient of the pressure is $\dfrac{dp}{dv}$, the volume and pressure gradients of the temperature are

$$\left(\frac{dt}{dv}\right) = t'_p.\frac{dp}{dv} + t'_v$$

and

$$\left(\frac{dt}{dp}\right) = t'_p + t'_v.\frac{dv}{dp}.$$

Or again, if p'_t and p'_v be the temperature and volume gradients of the pressure with volume and temperature respectively kept constant, then for any change defined by a volume-gradient of temperature $\dfrac{dt}{dv}$, the temperature and volume gradients of the pressure are

$$\left(\frac{dp}{dt}\right) = p'_t + p'_v.\frac{dv}{dt}$$

and

$$\left(\frac{dp}{dv}\right) = p'_t.\frac{dt}{dv} + p'_v.$$

Here p'_v is the slope of the "**isothermal**" on the p,v diagram; p'_t is the slope of the "**isometric**" on the p,t diagram; t'_v is the slope of the "**isobaric**" on the t,v diagram, etc., etc.

What is called "**Entropy**," usually symbolised by ϕ, is most simply defined by the equation of its increment

$$\delta\phi = \frac{\delta p.\delta v}{\delta t}$$

and this, combined with the above equations, gives most of the mathematical formulas of thermodynamics.

167. x-Gradient of (xy).—In fig. 17, § 80, the rectangular area between the two axes and the two co-ordinates x and X was taken as a function of these co-ordinates, and differentiated with respect to them. The problem was there considered in reference

to the ordinates to the particular curve shown in fig. 17, which may be regarded as similar to the particular curve s of § 161. If in fig. 17 we now regard x and X as the ordinates to any point in the whole field of the figure, they will then be independent variables. It will now be better to name the vertical ordinate y, as X is throughout this book used to indicate a function dependent on x. The area (xy) will be a function of these two independent variables. Applying the law of § 161 to this function, we have

$$\left\{ \frac{d(xy)}{dx} \right\} = \frac{\partial(xy)}{\partial x}\text{with } y \text{ constant} + \frac{\partial(xy)}{\partial y}\text{with } x \text{ constant} \times \frac{dy}{dx}$$

$$= y + x\frac{dy}{dx} = y + xy'$$

where the bracket { } indicates that the gradient is taken with concurrent change of x and y in the ratio given by y' on the right side. If the given y' be the x-gradient of the curve drawn in fig. 17, there is here reproduced the law of § 80, which is thus shown to be simply a particular case of a more general law, namely, that of § 161.

168. Definite Integral of Function of Independent Variables. —The equation of § 161 gives the increment of rise in level δh from any point of the surface to any other closely contiguous point. The integration of this increment of rise gives the total rise from one point to another point, near or distant, on the same surface. Taken between definite limits, this integral means the difference of level between two definite points on the surface. From any lower limit $n_1 w_1$ to any upper limit $n_2 w_2$ the definite integral is $(h_2 - h_1)$.

The indefinite integral is a general expression giving the height of the surface at any and every point measured from any convenient datum level.

169. Definite Integral of Function of Independent Variables. —In integrating from point 1 to point 2 (distant from each other), the integration may be followed out along a great variety of paths, the only condition a suitable path has to fulfil being that it must pass through both points 1 and 2. The path may be curved in any fashion, or be zigzagged in any regular or irregular manner. The integration along every such path will evidently give the same result. If in fig. 28 the points 1 and 2 be distant from each other, the integration might first follow the directly north path 1N, and then the directly west path N2. During the first part δw would be continuously zero, and the integration would extend from latitude n_1 to latitude n_2, keeping constantly to the longitude

w_1. During the second part, δn would be continuously zero. The same result is obtained by integrating first from 1 to W at constant latitude n_1, and then from W to 2 at constant longitude w_2.

170. Equation between Differences of Integrals.—Incidentally it may be noted that this gives, by converting the equation between the sums of these pairs of rises into an equation between the differences of the pairs of rises on opposite sides of the rectangle,

$$\left[\int_{n_1}^{n_2} \frac{\partial h}{\partial n} dn \right]_{w_1}^{w_2} = \left[\int_{w_1}^{w_2} \frac{\partial h}{\partial w} dw \right]_{n_1}^{n_2}$$

the left-hand expression meaning the difference between two integrations from latitude n_1 to latitude n_2, carried out along the meridians of longitude w_2 and w_1; while the right-hand similarly means the difference between two integrations each along a parallel of latitude and each between the same limits of longitude.

171. Indefinite Integral.—The indefinite integral h may be obtained by first integrating along any meridian up to an undefined point, and then from that same point along a parallel of latitude an indefinite distance; or the integration along the parallel of latitude may be effected first, to be followed by the meridional integration. In either case the second integration **must start from the same point as that at which the first finishes**, this point, however, being any whatever.

172. Independent Functional Integration Constants. — Although n and w may be varied quite independently, there is a relation between the law of the meridional section and that of the section of constant latitude which deserves notice. The equation of the meridional section, in which n is the variable, changes from section to section, *i.e.*, changes with the longitude. This equation, therefore, in general involves the longitude w. For any one such section the value of w entering into it remains constant. Thus the general expression for h may be taken as the sum of three terms, thus:—

$$h = N + F(n,w) + W$$

where N is a function involving n but not w; W a function involving w but not n; and $F(n,w)$ is the sum of such terms as involve both n and w. The partial gradient for any meridional section is

$$\frac{\partial h}{\partial n} = N' + F'_n(n,w)$$

W being a constant in this differentiation.

The partial gradient for any section of equal latitude is

$$\frac{\partial h}{\partial w} = W' + F'_w(n,w).$$

These two formulæ exhibit clearly the necessary relation between the two partial gradients. They differ, first, in N' and W', which are respectively functions of n alone and of w alone, and between which parts, therefore, there is complete independence; and, secondly, in $F'_n(nw)$ and $F'_w(nw)$, which are different but not independent, being necessarily related by the condition that they are the partial gradients of the same function involving both variables.

173. Independent Functional Integration Constants. — Written in terms of independent variables x, y, and χ the integral function of xy, these formulæ become

$$\chi = X + F(x,y) + Y$$

$$\frac{\partial \chi}{\partial x} = X' + F'_x(xy)$$

$$\frac{\partial \chi}{\partial y} = Y' + F'_y(x,y)$$

where X is a function of x only, and Y is a function of y only.

174. Complete Differentials.—In fig. 28 the slopes of the two lines 1W and N2 in the "North Elevation" give the westward gradient at the two latitudes n_1 and n_2. These lines are drawn parallel in fig. 28 because the points 1, 2 are close together, and for a **first degree of approximation** the difference of slopes through them may be neglected if the surface be continuous. If a second degree of approximation to accuracy be considered; that is, if we investigate "**second gradients**," the difference between these two westward gradients must be taken into account. It is evidently

$$\frac{d}{dn}\left(\frac{\partial h}{\partial w}\right) . \, \delta n$$

and the difference between the rise from N to 2 and the rise from 1 to W is

$$\frac{d}{dn}\left(\frac{\partial h}{\partial w}\right) . \, \delta n . \, \delta w .$$

Similarly, the difference between the northward gradients W2 and 1N as seen in the "West Elevation" of the same figure is

$$\frac{d}{dw}\left(\frac{\partial h}{\partial n}\right) . \, \delta w$$

and the difference between the rise from W to 2 and the rise from 1 to N is

$$\frac{d}{dw}\left(\frac{\partial h}{\partial n}\right).\,\delta w.\,\delta n\,.$$

But by § 170 these differences equal each other. Cancelling out the common product $\delta n.\delta w$, we have the equality

$$\frac{d}{dn}\left(\frac{\partial h}{\partial w}\right)=\frac{d}{dw}\left(\frac{\partial h}{\partial n}\right).$$

Using the nomenclature of the end of § 172, since $\dfrac{d\mathrm{W}'}{dn}=0$, because W' does not involve n, and similarly $\dfrac{d\mathrm{N}'}{dw}=0$, this equation becomes

$$\frac{d}{dn}\Big(\mathrm{F}'_{w}(nw)\Big)=\text{say }\mathrm{F}''_{nw}(nw)=\frac{d}{dw}\Big(\mathrm{F}'_{n}(nw)\Big)=\text{say }\mathrm{F}''_{wn}(nw)\,.$$

Thus it is indifferent whether the n or the w differentiation be taken first, and whether $\mathrm{F}''_{nw}(nw)$ or $\mathrm{F}''_{wn}(nw)$ be used as symbol.

Although these second-gradients, calculated in these two different ways, have the same value, they represent two perfectly distinct physical phenomena. The one is the *northward* rate of change of the *westward* gradient of h. The other is the *westward* rate of change of the *northward* gradient of h. That these are equal, whatever kinds of physical quantities be represented by h, n and w, is a proposition of mathematical physics that is most interesting and fertile in its various concrete applications.

175. Second x,y-Gradient.—When two functions of x and y fulfil the condition of being the partial x and y gradients of one and the same function, then the function formed by adding the products of these functions by δx and δy respectively, is said to be a **complete differential**. Thus if $\dfrac{\partial \chi}{\partial x}$ and $\dfrac{\partial \chi}{\partial y}$ be the functions, ascertained to be the partial x and y gradients of the same function χ, then

$$\frac{\partial \chi}{\partial x}.\,\delta x+\frac{\partial \chi}{\partial y}.\,\delta y$$

is a "**complete differential**," and this latter is said to be "**integrable**." If this has been found, by **accurate** deduction from correct observation of physical fact, to be the increment of a real

physical quantity, then it is certain that the function is theoretically, integrable (although the integration may be impracticably difficult) and that its two parts will fulfil the condition of § 174. Of course, it is easy for the pure mathematician to invent functions of this sort that are not integrable, and incorrect physical observation or inaccurate deduction from physical investigation may lead to differentials that are not integrable; but such have no real physical meaning.

176. Double Integration by dx and dy.—Conversely, if any function of two independent variables, x, y, be twice integrated first by dx and then by dy, the result will be the same as if first integrated by dy and then by dx, being in either case the sum of a function dependent on both x and y and of two other functions depending separately, one of them on x alone and the other on y alone.

These two latter functions are introduced by the integrations in the same way as constants are introduced by integrations with respect to one variable; the one function being a constant with respect to one variable, and the other being a constant with respect to the other variable.

Thus, for example, if

$$\frac{d^2\chi}{dx\,dy} = a + bx + cy + e\,xy$$

then

$$\chi = X + Y + xy\left(a + \frac{b}{2}x + \frac{c}{2}y + \frac{e}{4}xy\right)$$

where the laws of the functions X and Y must be determined by "limiting conditions."

The finding of χ from the given value of $\frac{d^2\chi}{dx\,dy}$ is called the double integration of this function, and is symbolised by

$$\iint \frac{d^2\chi}{dx\,dy}\,dx\,dy$$

or $\iint \phi(xy)dx\,dy$ if $\phi(xy)$ be the given functional form of $\frac{d^2\chi}{dx\,dy}$.

177. Graphic representation of Double Integration by dx and dy.—The meaning of the double integration of $\phi(xy)$ may be represented graphically in the following different manner.

Let $\phi(xy)$ be represented by the height of a surface from a datum plane, the co-ordinates parallel to this plane being x and y.

Then the first integration $\int \phi(xy)dx$ may be considered as extending along a section perpendicular to the datum plane and parallel to the x-axis, in this integration y being a constant. The result of this integration is a general formula for the area of any such section. Two such sections at the very small distance δy apart will inclose between them, under the surface and above the datum plane, a volume equal to δy multiplied by the area of the y-constant section at the middle of δy. This volume is, therefore, $\left\{ \int \phi(xy)dx \right\}.\ \delta y$, and the whole volume under the surface and above the datum plane therefore properly represents $\int\int \phi(xy)dxdy$. This geometric conception is more easily grasped if the integration be taken between limits.

178. Connection between Problems concerning One Independent Variable and those concerning Two Independent Variables.—In an investigation concerning two mutually dependent variables, such as those in Chapters I. to VII., the two variables may always be represented by the co-ordinates to a plane curve. This curve may be looked on as a plane section of a surface, the three co-ordinates to the points upon which are related to each other by the more general kind of law dealt with in this chapter. Thus the former problems may always be conceived of as the **partial** solutions of more general laws connecting three variables with only one specific relation between them. The problems of Chapters I. to VII. may thus be considered **special cases** of more general problems of the kind now dealt with, and each of them might be deduced by specialising from a more general theorem.

CHAPTER IX.

MAXIMA AND MINIMA.

179. General Criterions.—In fig. 1, at the parts C, E, H, K, R, S, U, the l-gradient of h is zero. The points C and K are places where h rises to a maximum, the maximum K being greater than the maximum C, but the phrase "**maximum**" being understood to mean a value greater than any neighbouring value on either side. E is a place where h falls to a minimum.

Thus the gradient falls to zero wherever there is either a maximum or a minimum value.

At the maxima points, C and K, the forward gradient passes through zero by changing from positive to negative, that is, the increase of the gradient is negative at these places.

At the minimum point E, the forward gradient changes from negative to positive, so that its increase is positive.

Thus the criterion for distinguishing between a maximum and a minimum is, that at the former the **second gradient** or second differential coefficient is negative, while at a minimum point it is positive.

It is not always, however, necessary to find the sign of the second gradient in order to make sure whether the point is a maximum or a minimum. For instance, if it be known that at the place where the first gradient is zero, the value of h is positive, and if it be also known that at two points near and on either side of this place the value of h becomes zero, or of any positively less value than at this place of zero gradient, then evidently this place gives a maximum.

At the place H, fig. 1, the second gradient is zero, because to the left of H it is negative, while to the right of H it is positive. This case of zero second-gradient occurring along with zero first-gradient is the limiting case coming in between the two previous ones, giving respectively maxima and minima; and it gives neither a maximum nor a minimum. This includes the case of the dead level RS, where also both first and second gradients are zero.

Usually one's general knowledge of the physical phenomenon being investigated is sufficient, without need of evaluating the second gradient, to indicate whether or not there is any such point as H. That is, the practical man who thinks of what he is working at, and does not follow blindly mere mathematical formulas, runs substantially no risk of mistaking such a point as H for either a maximum or a minimum point.

180. Symmetry.—In very many practical problems conditions of symmetry show clearly where a maximum or minimum occurs without the need of investigating either first or second gradient. Thus, if a beam be symmetrically supported, symmetrically loaded, and have a symmetrical variation of section on either side of a certain point of its length, which point is then properly called its centre, then the bending moment and the deflection each reach a maximum at this centre. Such considerations are to be utilised wherever possible, and their use is sometimes more profitable in practical result than the more strictly mathematical process.

181. Importance of Maxima in Practical work.—As examples

of the utility of these theorems may be cited the finding of the positions and magnitudes of maximum bending moments, of maximum stresses, of maximum deflections, of maximum velocities, of maximum accelerations of momentum, of the positions of rolling load on bridges to give maximum stress in any given member of the bridge, etc., etc. All these things are of special importance in the practical theory of engineering. In the jointing of pieces together in machines and static structures, it is never possible to obtain uniform stress over the various important sections of the joint. It is of the greatest importance to find the maximum intensities of stress on such sections, because the safety of the construction depends on the maximum, hardly ever upon the average, stress. The average stress on the section is found by dividing the whole load on the section by the whole area of the section. Such average stresses are often very different from the maximum stress, and no reliance ought to be placed upon them as measures of strength and safety.

Another class of technical problems in which maxima points are of paramount importance is that in which two or more sets of variable driving efforts, or of variable resistances, are superimposed in a machine. Thus a first approximation to the turning moment on the crank shaft of a steam engine of one cylinder, makes this moment vary as $\sin a$, where a is the angle at which the crank stands from the dead point. If there be two cylinders in which the total steam pressures are P_1 and P_2, constant throughout the stroke, and the two cranks, keyed on the same shaft, stand apart by an angle A; then a being the angle from dead point of one crank, $(a + A)$ is that of the other. A remains constant while a varies. If S_1 and S_2 be the two strokes, the total turning moment on the shaft is

$$\tfrac{1}{2}\{S_1 P_1 \sin a + S_2 P_2 \sin (a + A)\}$$

which reaches a maximum when its a-gradient is zero; that is, when

$$\frac{\cos a}{\cos(a + A)} = -\frac{S_2 P_2}{S_1 P_1}.$$

This ratio is *minus* unity when $S_2 P_2 = S_1 P_1$; and if, further, $A = 90°$, then $a = 45°$ at the maximum.

182. Connecting Rod Bending Moments.—The connecting rod of an engine is at each instant bent by transverse accelerations of momentum, which, taken per inch length, would increase uniformly from zero at the crosshead to a certain amount at the crank end if the section of the rod were uniform. The

actual bending moments on the rod follow nearly the law due to this distribution of load, because the excess of weight in each head is approximately centred at the point of support at either end and, therefore, does not affect the bending moments.

If L be the whole length; l the length to any section from the crosshead; w the transverse load per inch at the crank end: then $\dfrac{wl}{L}$ is the load per inch at l. On the section at l, therefore, the bending moment is $\left\{ \dfrac{wL}{6} \cdot l - \dfrac{wl^2}{2L} \cdot \dfrac{l}{3} \right\}$.

The first l-gradient of this is zero at the point of maximum moment; that is, this point has a distance l given by

$$\frac{wL}{6} - \frac{wl^2}{2L} = 0$$

or

$$l = \frac{L}{\sqrt{3}} = \cdot 5773L.$$

Inserting this value of l in the general value of the moment, we find as the maximum moment

$$\left\{ \frac{wL}{6} \cdot \frac{L}{\sqrt{3}} - \frac{wL^3}{L6 \times 3\sqrt{3}} \right\} = \cdot 06415wL^2$$

which may be compared with $\cdot 0625wL^2$, which is the central moment in the case of the same total load, $\frac{1}{2}wL$ being uniformly distributed along the whole span. It is $2\frac{1}{2}\%$ greater than this latter, and its position is $7\frac{1}{2}\%$ of the span away from the centre.

183. Position of Supports giving Minimum Value to the Maximum Bending Moment on a Beam.—The following illustrates how maxima of arithmetic, as distinguished from algebraic, quantities may sometimes be found without use of a differential coefficient. If a beam, freely supported, overhang its supports equally at the two ends, and be uniformly loaded; then certain positions for the supports will make the maximum moment less than for any other positions of these supports.

Let L and w per inch be the total length and the load, and l the span between the supports. The bending moment on the section over each support is $w \cdot \dfrac{L-l}{2} \cdot \dfrac{L-l}{4} = \dfrac{w}{8}(L-l)^2$. The central moment, taking it of opposite sign, is

$$\frac{wL}{2} \cdot \frac{l}{2} - \frac{wL}{2} \cdot \frac{L}{4} = \frac{wL}{4}\left(l - \frac{L}{2}\right).$$

If this latter be negative, *i.e.*, if $l < \dfrac{L}{2}$, these two moments will be of the same physical sign; that is, the beam will be bent convex on its upper surface throughout its whole length. If $l > \dfrac{L}{2}$, a certain central length will be concave on the upper surface, and inside and outside this length the moments will be of opposite sign. As l is made larger, the magnitude of the central moment becomes always larger and that of the moment at the supports always smaller. Therefore, neither has any algebraic maximum. But when they are arithmetically equal, their common arithmetic magnitude is then less than the magnitude of the greater of the two for any other span. So that, irrespective of sign, the minimum of the arithmetic magnitudes of the three maximum moments is reached when

$$\frac{w}{8}\left(L - l\right)^2 = \frac{wL}{4}\left(l - \frac{L}{2}\right)$$

or

$$l = \cdot 5858L \text{ and } \frac{L - l}{2} = \cdot 2071L.$$

Inserting this value of l in either formula for the moment we find

Central moment $=$ moment over each support $= \cdot 02144 wL^2$

which is only 17% of the central moment on the same beam with same load when supported at the two ends. This fact may be regarded as the basis of the great economy of the modern "canti-lever" style of bridge building.

184. Position of Rolling Load for Maximum Moment and for Maximum Shear.—The next example shows how reasoning about increments, instead of differential coefficients, may be used to find maximum values.

The bending moment produced on any section of a girder by a load is of the same sign wherever the load be placed within the span. Therefore the moment on each and every section produced by a uniform rolling load reaches a maximum when the load covers the whole span.

The right-handed integral shear stress on each section equals the supporting force at the left-hand support, *minus* the load applied between this support and the section. Therefore, any load applied right of the section increases this shear stress, because it increases the left supporting force and leaves unaltered the load between it and the section. But a load applied left of the section decreases the same stress, because it increases the left supporting force less

than it increases the load between it and the section. Therefore the right-handed shear stress on any section due to a uniform rolling load reaches a maximum when the load covers the whole of that part of the span to the right of the section, but covers none to the left of it. The left-handed shear stress reaches its maximum when the part left of the section is covered. The arithmetic maximum of the stress is reached when the larger of the two segments into which the section divides the span is covered while the shorter is empty.

185. Most Economical Shape for I Girder Section. — The economic proportioning of sections is illustrated by the following.

Let M be the bending moment strength of an I girder, whose depth is H outside the flanges and h inside them, and whose flange breadth is B and web thickness wB.

Let the area of its cross-section be called S.

Then the moment strength per square inch of section may easily be shown to be

$$\frac{M}{S} = \frac{k}{6H} \cdot \frac{H^3 - (1 - w)h^3}{H - (1 - w)h}.$$

If H and h be increased in the same proportion, this $\dfrac{M}{S}$ will increase in proportion to the first power of either of them, large sections being always stronger and stiffer per square inch than small ones. It also increases if H is increased without alteration of h. $\dfrac{M}{S}$ also increases as w is decreased towards zero, the web section contributing to the moment strength less than the flange section and, therefore, less than the average for the whole section. If, however, the web thickness be supposed fixed in accordance with the requirements of shear strength, and if h be diminished while H is unaltered, thereby thickening the flanges *internally*, this flange thickening will, up to a certain limit, increase the economy of the section, beyond which a further thickening will decrease it again. The h-gradient of $\dfrac{M}{S}$ is

$$\frac{k}{6H} \cdot \frac{-3(1 - w)h^2\{H - (1 - w)h\} + (1 - w)\{H^3 - (1 - w)h^3\}}{\{H - (1 - w)h\}^2}.$$

If this be equated to zero, there results

$$2(1 - w)\left(\frac{h}{H}\right)^3 - 3\left(\frac{h}{H}\right)^2 + 1 = 0.$$

This is a cubic equation giving the most economical depth inside

the flanges when that outside the flanges, as also the ratio w of web thickness to flange width, are fixed by other considerations. This ratio between h and H essentially depends on w; if $w = 0$, giving zero thickness to the web, the above equation gives $h = H$, *i.e.*, gives zero thickness to the flange also, or the whole section shrinks to zero area. When $w = \cdot5$, it gives $h/H = \cdot6527$. A useful exercise for the engineering student is to solve this equation for values of w ranging up to $\cdot5$. The solution can be very easily effected by the method of solving for w taking a series of values of h/H ranging from 1 down to $\cdot6$; tabulating these graphically as a curve; and then reading from the curve the h/H for any desired values of w.

186. Most Economical Proportions for a Warren Girder.— The economic proportioning of general dimensions is the subject of the next example.

If a Warren girder of height H, and length of bay B, have the bay width B made up of b the horizontal projection of a tie-brace, and $(B - b)$ the horizontal projection of a strut brace; then the weight of material G required to give the structure strength to carry the desired load, exclusive of that spent in jointing the various members together, may be expressed by a formula of six terms involving, besides H, B and b, also the span, the load, the stresses allowed on the sections, and four numerical coefficients which do not vary with the H nor with the span or load nor with the ratio $\dfrac{b}{B}$ and vary very little with the number of bays. The same formula may be applied to any pattern of lattice girder by suitably adjusting the numerical coefficients.

Four terms of this weight decrease as H increases, while two increase. A certain girder depth H will, therefore, be most economical in expenditure of material. Assuming everything but H to remain constant, and equating the H-gradient of the above to zero, there is obtained the best girder depth.

Again, the girder weight contains two terms increasing with B and two others decreasing with B. Assuming the ratio $\dfrac{b}{B}$ and all other quantities except B to be kept unaltered and equating the B-gradient of G to zero, we obtain a formula for the most economical bay width for given span and height, which gives also indirectly the best number of bays to insert in the given span. This formula cannot, however, be precisely followed, because the number of bays must necessarily be a whole number while the equation gives in general a fractional number.

The girder weight also varies with b in two terms, one of which

increases while the other decreases with b. Considering everything but b as fixed, and putting the b-gradient equal to zero, a rule is found for proportioning the length of the ties to that of the struts.

These results are not formulated here because to guard against their incorrect application requires rather more explanation of special bridge-building detail than is suitable to this treatise.

187. Minimum Sum of Annual Charge on Prime Cost and of Working Cost.—Very many technical problems are, or ought to be, solved by reducing to a minimum the sum of two main costs : first, the initial cost of construction and other necessary preliminary expenses ; second, the cost of working, maintenance, and repair. These can only be added when reduced to terms rationally comparable, and this is usually done by reducing both to an **annual** cost or charge. Interest on all initial expenses, including prime cost of actual construction, is to be added to an annual charge to provide for a sinking fund to reproduce the capital after a period within which it is estimated that the plant will become useless from being worn out, or having become obsolete —which annual charge is often referred to as "**depreciation**"— and this forms the first part of the whole cost. The second part consists of wages, materials used up in working, power for driving, etc. If the initial expenditure be skilfully and wisely spent, its increase nearly always, within limits, decreases the working expenses. It follows that in most if not all cases a certain initial expenditure is that that will make the total annual cost a minimum. Thus the adoption of a larger ratio of expansion in a steam engine will, within certain limits, diminish the consumption of water and of coal required to produce any required horse-power ; but it will necessitate a larger and more expensive engine for this same horse-power, which will be, moreover, more costly to keep in good working order ; and this is the real consideration which ought to determine the commercially most economic cut-off in steam engines. Lord Kelvin's calculation of the best cross-sectional area of electrical leads is another example of this kind of problem.

188. Most Economical Size for Water Pipes.—The following is a similar example directed to the calculation of the most economic diameter of water pipes, first published by the author in 1888.

If a given weight or volume of water is to be delivered per hour at a certain station at a certain pressure, this means the same thing as delivering so much water horse-power at this station. Let this horse-power be called H, and the pressure demanded at the point of delivery p ; let L be the distance from the pumping or gravity-

power station, and d be the internal diameter of the pipe. Then the loss of power in transmission, through friction and viscosity (exclusive of loss at bends and valves), can be shown to be nearly $a\dfrac{LH^3}{p^3d^5}$, where a is a numerical coefficient dependent on the smoothness of the inside surface of the pipe, and on the shape of cross section. If q be the cost per hour of generating 1 horse-power, and if the delivery be continued for T hours per year; then the cost of this waste horse-power per year is

$$q\mathrm{T}a \cdot \frac{LH^3}{p^3d^5}.$$

The prime cost of pipes and pipe-laying (including trenching) may be taken as the sum of two terms, the first proportional to the length L, and independent of the size of pipe; the second proportional to the quantity of metal in the pipe. The thickness of pipe requires to be designed according to the formula $\left(A + \dfrac{pd}{B}\right)$ where A and B are constants. The part of the initial cost which varies with the diameter will, therefore, give an annual cost in interest and depreciation of

$$r\mathrm{L}d\left(A + \frac{pd}{B}\right)$$

where r is a factor dependent on (1) the price of iron; (2) the nature of the ground to be trenched; and (3) the prevailing rate of interest on money.

That part of the total annual cost which varies with the size of the pipe is, therefore,

$$q\mathrm{T}a\frac{LH^3}{p^3d^5} + r\mathrm{L}d\left(A + \frac{pd}{B}\right).$$

Equate the d-gradient of this to zero, and there results the equation

$$\left(A + 2\frac{pd}{B}\right)d^6 = 5 \cdot \frac{qa}{r} \cdot \mathrm{T} \cdot \left(\frac{H}{p}\right)^3$$

an equation for the determination of d, which, in spite of its being of the 7th degree, is very easily solved by a few trials carried out with the help of Barlow's Tables of powers. Other things being

equal, it makes the diameter vary according to a power of $\dfrac{H}{p}$ lying between $\frac{1}{2}$ and $\frac{3}{7}$.*

189. Maximum Economy Problems in Electric Transmission of Energy.—Calculations of maxima enter into many important problems in electric engineering. Let an electric current generator exert an E.M.F. equal to E, and that part of the electric resistance of the circuit under the control of the supply company be R, while the rest of the circuit has a resistance r. Then the current is $\dfrac{E}{R+r}$ if there be no counter E.M.F. The electric work per second in the external resistance r is $\dfrac{E^2 r}{(R+r)^2}$. If it be desired to make this as great as possible by adjusting r without alteration in either E or R, we find the r-gradient of this external work to be zero when $r = R$, and the external work is then $\dfrac{E^2}{4r}$, the whole work being double this.

If there be an external counter E.M.F. equal to e, the current is $\dfrac{(E-e)}{R+r}$ and the work done on the counter E.M.F. is $\dfrac{(E-e)e}{R+r}$. This work continuously decreases as either r or R is increased without change in E or e. If E, R, and r be fixed while e is adjustable, the e-gradient of this work is $\dfrac{E-2e}{R+r}$, which becomes zero when $e = \frac{1}{2}E$, giving the maximum work that can be done on e under these conditions equal to $\dfrac{E^2}{4(R+r)}$ and the efficiency $\dfrac{e}{E}$ equal to $\frac{1}{2}$. This gives maximum motor work under the prescribed conditions, *not maximum efficiency.*

In the early days of attempted transmission of power by electric current, it was believed by many that this showed that no higher efficiency than $\frac{1}{2}$ could be reached in such transmission. This misapprehension was acted upon practically by electricians of good reputation, and retarded for many years the progress of this branch of engineering.

The efficiency of the transmission is $\dfrac{e}{E}$, and this can be made nearly unity by making e nearly equal to E. By so doing, the

* After demonstrating this law, the author incidentally discovered that an old established firm of water engineers had empirically framed a rule which was nearly identical with the above.

H

excess of driving over driven E.M.F., namely $(E - e)$, is diminished, and the current and horse-power delivered are diminished unless the decrease so effected be neutralised by increase in e, and, therefore, also in E, or by decrease in $R + r$. With small resistances, however, combined with high voltages, large horse-powers can be transmitted with high electric efficiency. The necessary voltages and resistances required to deliver any required horse-power with any given efficiency are easy to calculate, but the calculations do not illustrate the subject of this chapter.

A large number of electric transmission calculations of values giving maximum economy, etc., under various conditions, may be found in a series of articles by the author in *Industries*, 1889.

190. Maxima of Function of Two Independent Variables.— If a function of two independent variables be represented by the height of a surface with the two variables as horizontal co-ordinates, then any plane vertical section will give a curve which will rise to maximum height or fall to minimum height where the partial first gradient is zero, and the partial second gradient is not zero. If two such sections cross each other at a surface point where the partial first gradients are zero in both sections, then the surface is level throughout a small extent all round the intersection point. There are six cases to be distinguished.

(1) The partial second gradient is positive in both sections; then the point is at the bottom of a hollow in the surface, all neighbouring points being higher; so that the value of the function represented by the height is here a minimum.

(2) The partial second gradients in the two sections are both negative; then the point is at the top of a convexity or globular part of the surface, all neighbouring points being lower; so that the value of the height function is here a maximum.

(3) The one partial second gradient is positive, while the other is negative. Here the surface is anticlinal, or saddle-shaped; it is hollow in one direction, and round in the other. This gives neither minimum nor maximum value to the height function.

(4) The one partial second gradient is positive, while the other is zero, passing from positive to negative. Here the part of the surface is the junction between a hollow portion lying on one side of a vertical plane and an anticlinal portion lying on the other side of the same; and the height function is again at neither maximum nor minimum value.

(5) The one may be negative, while, as in (4), the other is zero.

There is here indicated the junction between a round portion and an anticlinal portion; and again neither maximum nor minimum occurs here.

(6) Again both may be zero. Here two semi-anticlinal surfaces join together, and the point gives neither maximum nor minimum.

The first two cases alone are important as regards finding maximum and minimum values. In these cases both first gradients are zero, and the second gradients are of the same sign. To find the maximum or minimum values of a function of two independent variables, the process is to equate the two partial first gradients to zero, and combine these two equations as simultaneous ones. Afterwards the second gradients should be examined as to sign if the physical character of the problem is not so plain as to make this unnecessary.

191. Most Economic Location of Junction of Three Branch Railways.—As an illustration of this process, take an elementary problem in the theory of railway location. Three centres of traffic are supposed situated in a plain across which the construction of the railway is equally easy in all directions. The three centres are to be joined by three branch lines radiating from a junction, the finding of the best position for which junction is the problem proposed. The traffic issuing from and entering each centre is the sum of the two traffics between it and the other two (inclusive possibly of traffic going *through* these to more distant points). The importance of the three traffics to and from the three centres being properly measured, and being here symbolised by A, B, and C; and the distances of the three centres from the junction being called a, b, c; the correct solution of this problem means the location of the junction so as to give a minimum value to

$$Aa + Bb + Cc$$

A, B, C being given, while a, b, c are to be found.

Referring to the notation of fig. 29, in which the distance between the points of traffic A and C is called L, and the projection on L of the distance between the points of traffic A and B is called λ, while the projection of the same perpendicular to L is called H; the problem may be conveniently stated to be to find the co-ordinates l and h of the best junction.

We have with this notation

$$a = \{l^2 + h^2\}^{\frac{1}{2}} \qquad b = \{(\lambda - l)^2 + (H - h)^2\}^{\frac{1}{2}} \qquad c = \{(L - l)^2 + h^2\}^{\frac{1}{2}}.$$

The two partial l and h gradients of $(Aa + Bb + Cc)$ are taken; thus

$$\frac{\partial a}{\partial l} = \frac{l}{(l^2 + h^2)^{\frac{1}{2}}} \quad = \frac{l}{a} \quad = \quad \cos a$$

$$\frac{\partial b}{\partial l} \quad\quad = -\frac{\lambda - l}{b} = \quad -\cos \beta$$

$$\frac{\partial c}{\partial l} \quad\quad = -\frac{L - l}{c} = \quad \cos \gamma$$

$$\frac{\partial a}{\partial h} = \frac{h}{(l^2 + h^2)^{\frac{1}{2}}} \quad = \frac{h}{a} \quad = \quad \sin a$$

$$\frac{\partial b}{\partial h} \quad\quad = -\frac{H - h}{b} = \quad -\sin \beta$$

$$\frac{\partial c}{\partial h} \quad\quad = \frac{h}{c} \quad = \quad \sin \gamma$$
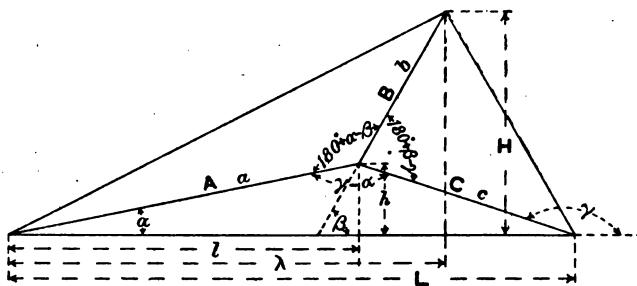
Fig. 29.

Therefore equating both partial gradients to zero, we obtain the simultaneous equations

$$\left.\begin{array}{l} A \cos a - B \cos \beta + C \cos \gamma = 0 \\ A \sin a - B \sin \beta + C \sin \gamma = 0 \end{array}\right\}.$$

Eliminating the terms in B by cross-multiplication, and again similarly eliminating the terms in C, there are obtained the two ratios

$$\frac{A}{C} = \frac{\sin (\gamma - \beta)}{\sin (\beta - a)} = \frac{\sin (180° + \beta - \gamma)}{\sin (180° + a - \beta)}$$

and

$$\frac{A}{B} = \frac{\sin (\gamma - \beta)}{\sin (\gamma - a)} = \frac{\sin (180° + \beta - \gamma)}{\sin (\gamma - a)}.$$

By reference to the figure it will be seen that the three angles entering into these ratios, namely, $(180° + \beta - \gamma)$, $(180° + a - \beta)$,

and $(\gamma - a)$, are the three angles between b and c, a and b, c and a. Since the sides of a triangle are proportional to the sines of the opposite angles, it follows immediately that these three angles $\overset{\wedge}{bc}$, $\overset{\wedge}{ab}$, and $\overset{\wedge}{ca}$, to which the three branches are to be adjusted, are the exterior angles of a triangle whose sides are made equal (to any convenient scale) to A, C, and B. By constructing this triangle these angles can be found, and by drawing upon two of the lines joining two pairs of centres of traffic two arcs of circles containing these angles, the proper junction is located as the intersection of these arcs.

This result may be perceived more directly by noticing that the two partial gradients to be equated to zero are, one the sum of the projections on L, and the other the sum of the projections on H, of A, B, C, measured along the lines a, b, c, outwards from the junction.

CHAPTER X.

INTEGRATION OF DIFFERENTIAL EQUATIONS.

192. Explicit and Implicit Relations between Gradients and Variables.—The utility of the art of integration arises from the fact that in the investigation of phenomena it often happens that the discovery of the ratio between simultaneous increments (or gradient) of mutually related quantities is effected more easily by direct observation than is the discovery of the main complete relation between these same quantities. This complete relation is then logically deduced by the help of integration, combined with the further observation of special or "limiting" values. The complete relation being less general than the differential relation, there appears in it an "**arbitrary constant**" whose value is not given by the differential relation, and which value must be discovered by examination of the "**limiting conditions.**" The differential relation applies equally well to a whole "**family**" of integral relations which differ among themselves in respect of these limiting conditions.

If, when the differential relation is expressed as an equation, the gradient can be placed alone on one side of the equation, while on the other appears a function of **one only** of the mutually dependent variables; then, in order to establish the integral relation, nothing

more has to be done than to integrate this latter function **directly** according to one or other of the methods explained in previous chapters, or given in the appended *Classified Reference List of Integrals*. This is the case of the gradient being expressed as an **explicit function of one of the variables.**

If, however, the differential equation involve the gradient and both variables in such a way that the above simple separation does not appear; that is, if the gradient appear as an **implicit** function of the variables; then either the implicitness of the relation must be got rid of by some algebraic process, or else a special method of integration must be employed. This process is called the "**solution**" of the **differential equation.**

193. Degree and Order of an Equation. Nomenclature.—An ordinary algebraic equation is said to be of the nth **power** or **degree** when it involves the nth power of the "**unknown**" quantity or of either of the "**variables.**" In a differential equation, the gradients being the quantities to be got rid of by integration, the equation is said to be of the nth **degree** when the nth power of the highest occurring gradient is involved in it.

If it involve a **second gradient** $\left(\dfrac{d^2X}{dx^2}\right)$, it is said to be of the **second order.** If it involve an nth gradient, it is said to be of the nth **order.**

The integral relation deduced from the differential equation may be called the **integral equation,** but is also called the **primitive equation.**

In what follows X' will mean $\dfrac{dX}{dx}$;

and $f'(x) \equiv \dfrac{df(x)}{dx}$; $f'(X) \equiv \dfrac{df(X)}{dX}$ — not $\dfrac{df(X)}{dx}$; $f'(X') \equiv \dfrac{df(X')}{dX'}$.

194. X' = $f(x)$.

The form of differential equation simplest to integrate is the **explicit** relation,

$$X' = f(x)$$

where $f(x)$ is a function involving the variables x alone, and not X. This is equivalent to

$$dX = f(x)dx$$

and if $f(\ )$ is integrable by help of any of the formulas in the Reference List, the integration can be effected at once; thus:—

$$X = \int f(x)dx + C$$

where C is the constant of integration.

195. $X' = f(X)$.

Again, if $\qquad X' = f(X)$

where $f(X)$ involves X only and not x, then

$$\frac{dX}{f(X)} = dx$$

and direct integration of $\dfrac{1}{f(\ )}$, if it be possible by one of the formulas in the Reference List, gives the integral relations between X and x

$$\int \frac{dX}{f(X)} = x + C.$$

As an example, if $f(X) = X$; then $\log_e X = x + C$.

As another example, if $f(X) = \sin X$; then $\log_e \tan \dfrac{X}{2} = x + C$.

196. $X' = f(x)F(X)$.

Again, if the differential relation be found in the form

$$X' = f(x)F(X)$$

where $f(x)$ is a function involving x and not X,
while F(X)   ,,   ,,   X   ,, x;
then

$$\frac{dX}{F(X)} = f(x)dx$$

and

$$\int \frac{dX}{F(X)} = \int f(x)dx + C$$

which gives the integral solution if both these integrals can be effected directly.

As an example, take

$$X' = \cos x \sin X;$$

then

$$\log_e \tan \frac{X}{2} = \sin x + C.$$

The case of § 195 is only a special case of the more general method of the present article, namely, the case in which $f(x) = 1$; while that of § 194 is the other special case in which $F(X) = 1$.

In all these cases X' has been found as a function of x and X in a form in which X', x, and X can be completely separated as distinct terms or factors in the equation.

197. $x = f(X')$.

Suppose now that a certain function of the gradient X' is found to equal x; thus,

$$x = f(X').$$

This equation may be capable of easy algebraic solution so as to give X' as an explicit function of x; thus,

$$X' = f^{-1}(x)$$

where $f^{-1}(\ )$ means the function that is the **inverse** of $f(\ )$. It must be noted that such a solution has in general more than one root. Thus, if $f(X')$ be a quadratic rational function of X', there are two roots. This solution can then be dealt with by § 194; thus,

$$X = \int f^{-1}(x)dx + C$$

provided the function $f^{-1}(\ )$ turns out to be directly integrable.

This method, however, may be impracticable, or else may involve more labour than the following.

Take the X-gradient of

$$x = f(X')$$

$$\frac{dx}{dX} = \frac{1}{X'} = f'(X')\frac{dX'}{dX}$$

from which we deduce by transposition

$$dX = X'f'(X')dX'$$

and

$$X = \int X'f'(X')dX' + C.$$

Here $\int X'f'(X')dX'$ is the same function of X' as $\int xf'(x)dx$ would be of x; and if $xf'(x)$ is directly integrable by dx, the above can be directly found as a function of X'. Let this function be called $\phi(X')$, and suppose it expressed in terms of X'. Then from the two simultaneous equations, of which the first is the original differential equation,

$$\left.\begin{array}{c} x = f(X') \\ \text{and} \quad X = \phi(X') + C \end{array}\right\}$$

X' can be eliminated so as to leave an equation involving only x and X. This is the integral solution of the given differential equation.

As an example let
$$x = \sin X'.$$
Then
$$f'(X') = \cos X'$$
and
$$\int X' \cos X' dX' = X' \sin X' + \cos X'$$
$$= xX' + (1 - x^2)^{\frac{1}{2}} = x \sin^{-1} x + (1 - x^2)^{\frac{1}{2}}.$$
· Therefore
$$X = x \sin^{-1} x + (1 - x^2)^{\frac{1}{2}} + C.$$

The same result is obtained by solving the given differential equation for X', viz., thus $X' = \sin^{-1} x$, and integrating directly from this.

· **198.** $X = f(X')$.

If the implicit relation is found in the form
$$X = f(X')$$
either the algebraic solution of this for X' may be obtained, whence the integration
$$\int \frac{dX}{f^{-1}(X)} = x + C;$$

or else a method similar to that of last paragraph may be followed. Thus, taking the x-gradient,
$$X' = f'(X') \frac{dX'}{dx};$$
whence
$$\int \frac{f'(X')}{X'} dX' = \text{say } \phi(X') = x + C$$

this integration by dX' involving X' only, and giving some function of X', which is here symbolised by $\phi(X')$.

From the two simultaneous equations
$$X = f(X')$$
and
$$x + C = \phi(X')$$

X' is to be eliminated by ordinary algebraic means, leaving the integral equation involving only x and X.

199. $mX = X'$.

A particular case of the last is that of $f(X') = \dfrac{1}{m} X'$. Here

$$X = \frac{1}{m}X' = \frac{1}{m}\frac{dX}{dx}$$

or

$$dx = \frac{1}{m}\frac{dX}{X}$$

the integration of which gives directly

$$x = \frac{1}{m}\log_e(CX).$$

In fact, in this case the differential formula of § 198 reduces to that of § 195, and is the first of the two examples of the result of § 195 given in that paragraph.

200. $X = xf(X')$.

A differential formula only slightly different from that of § 198, and to be dealt with in the same general manner, is

$$X = xf(X').$$

Taking the x-gradients of both sides,

$$X' = f(X') + xf'(X')\frac{dX'}{dx}$$

from which

$$\frac{f'(X')}{X' - f(X')}dX' = \frac{dx}{x}$$

and therefore

$$\int \frac{f'(X')}{X' - f(X')}dX' = \text{say }\phi(X') = \log(Cx).$$

From the two simultaneous equations

$$\left.\begin{array}{c} X = xf(X') \\ \log(Cx) = \phi(X') \end{array}\right\}$$

and

X' is to be eliminated algebraically so as to leave the integral equation between x and X.

201. $X = nxX'$.

A particular case of the last formula, which is also at the same time a particular case of § 196, is

$$f(X') = nX'$$

or

$$X = nxX'$$

or

$$\frac{dX}{X} = \frac{1}{n}\frac{dx}{x}$$

whence by direct integration of each side

$$X = Cx^{\frac{1}{n}}.$$

202. $X = \pm xX' + f(X')$.

If the differential equation be of the more involved form

$$X = xX' + f(X'),$$

by taking the x-gradients on both sides, there is obtained, X' cancelling out from the two sides,

$$0 = \{x + f'(X')\}X''.$$

This equation has two solutions. The first is

$$X'' = 0$$

whence

$$X' = C_1 \text{ and } X = C_1 x + C_2$$
$$= C_1 x + f(C_1)$$

since $X = xX' + f(X')$ and $X' = C_1$. This is a **partial integration** of the given differential equation.

The other solution is

$$x + f'(X') = 0 \left.\right)$$

From this and

$$X = xX' + f(X') \left.\right)$$

treated as simultaneous equations, X' may be algebraically eliminated, leaving an equation giving X in terms of x either explicitly or implicitly. Let this equation be symbolised by $\phi(x, X) = 0$. This $\phi(x, X) = 0$ is a second **partial** solution of the given differential equation.

The combination of these two partial solutions gives the complete solution in its most general form, which, in application to whatever physical problem may be in hand, must be particularised by the insertion of the "limiting conditions." These limiting conditions sometimes exclude one of the "partial" solutions as impossible, leaving the other partial solution as the full true solution of the particular physical problem in hand.

The solution of a more generalised form of this differential equation is given in § 210, the method of solution depending on that of § 208.

A form differing from the last only in the sign of xX' is

$$X + xX' = f(X').$$

Here $X + xX'$ is the x-gradient of xX.

Therefore, the integration gives

$$x\mathrm{X} = \int f(\mathrm{X}').dx = \int \frac{f(\mathrm{X}')}{\mathrm{X}''}.d\mathrm{X}';$$

and, if X'' be expressible in terms of X' alone and the function $\frac{f(\mathrm{X}')}{\mathrm{X}''}$ be integrable by $d\mathrm{X}'$, this integration will give an equation between x, X, and X', between which and the original equation, X' may be algebraically eliminated, leaving one involving only x and X.

203. Homogeneous Rational Functions. — If the relation between x, X, and X' be found in the form

$$(ax^m + bx^{m-1}\mathrm{X} + cx^{m-2}\mathrm{X}^2 + \text{----})\mathrm{X}' = \mathrm{A}x^m + \mathrm{B}x^{m-1}\mathrm{X} + \mathrm{C}x^{m-2}\mathrm{X}^2 + \text{----}$$

where the (x,X) functions on the two sides are both "**homogeneous**" of the mth degree, that is, where each consists of a series of products of powers of x and X, the sum of the two indices in each term being m; then by dividing each side by x^m, this may be converted into an equation in $\dfrac{\mathrm{X}}{x}$.

Call $\dfrac{\mathrm{X}}{x} \equiv \mathcal{X}$, or $\mathrm{X} \equiv x\mathcal{X}$; therefore $\mathrm{X}' = \mathcal{X} + x\mathcal{X}'$.

Dividing by x^m, the differential equation becomes

$$(a + b\mathcal{X} + c\mathcal{X}^2 + \text{----})\,(\mathcal{X} + x\mathcal{X}') = \mathrm{A} + \mathrm{B}\mathcal{X} + \mathrm{C}\mathcal{X}^2 + \text{----}.$$

From this is easily deduced

$$\frac{dx}{x} = \frac{a + b\mathcal{X} + c\mathcal{X}^2 + \text{-------}}{\mathrm{A} + (\mathrm{B}-a)\mathcal{X} + (\mathrm{C}-b)\mathcal{X}^2 + \text{-----}}.d\mathcal{X}.$$

The integral of the left side is $\log x$. Therefore, if that on the right is directly integrable [*] to a function of $\mathcal{X}$, say $\phi(\mathcal{X}) \equiv \phi\!\left(\dfrac{\mathrm{X}}{x}\right)$; then the equation

$$\log x + \mathrm{C} = \phi\!\left(\frac{\mathrm{X}}{x}\right)$$

gives the desired integral relation between x and X.

A convenient shorthand symbol for such a homogeneous (x, X) – function of the mth degree is $f(x^{m-r}, \mathrm{X}^r)$. The two such functions

---

[*] See Classified List, III. A. 14.

may be called $f(x^{m-r}, X^r)$ and $F(x^{m-r}, X^r)$. The given differential equation may then be written

$$X'f(x^{m-r}, X^r) = F(x^{m-r}, X^r).$$

Dividing $f(x^{m-r}, X^r)$ by x^m we obtain the same function of 1 and $\mathfrak{X}$ as $f(x^{m-r}, X^r)$ is of x and X. The quotient may, therefore, be written $f(1^{m-r}, \mathfrak{X}^r)$, and similarly that of $F(x^{m-r}, X^r)$ by x^m may be written $F(1^{m-r}, \mathfrak{X}^r)$.

The integral equation then appears as

$$\log x + C = \int \frac{d\mathfrak{X}}{\dfrac{F(1^{m-r}, \mathfrak{X}^r)}{f(1^{m-r}, \mathfrak{X}^r)} - \mathfrak{X}}.$$

204. Homogeneous Rational Functions.—The last form is a particular case of a more general one involving the first power of X' only, and the **ratio only** of X to x. Call this ratio $\mathfrak{X}$ as in last article, so that as before $X' = \mathfrak{X} + x\mathfrak{X}'$. The present more general form of differential equation may be written

$$X' = f(\mathfrak{X}) = \mathfrak{X} + x\mathfrak{X}'$$

where $f(\ )$ indicates **any** form of function.
Therefore

$$\mathfrak{X}' = \frac{f(\mathfrak{X}) - \mathfrak{X}}{x}$$

or

$$\frac{dx}{x} = \frac{d\mathfrak{X}}{f(\mathfrak{X}) - \mathfrak{X}}.$$

The integral solution of this, namely,

$$\log x + C = \int \frac{d\mathfrak{X}}{f(\mathfrak{X}) - \mathfrak{X}}$$

gives x in terms of $\mathfrak{X} \equiv \dfrac{X}{x}$, and therefore gives X also in terms of x.

205. $X = xf(X')$.

A form of differential equation of cognate character is that solved in § 200, namely,

$$X = xf(X')$$

or

$$\mathfrak{X} = f(X') \text{ where } \mathfrak{X} = \frac{X}{x}.$$

One solution of this is given in § 200. Otherwise, it may possibly be more easily solved algebraically so as to give X' explicitly in

terms of $\mathfrak{X}$. Let $f^{-1}(\ )$ denote the inverse of the function $f(\ )$. Then this algebraic solution would appear as

$$X' = f^{-1}(\mathfrak{X})$$

of which, according to last article, the integral solution is

$$\log x + C = \int \frac{d\mathfrak{X}}{f^{-1}(\mathfrak{X}) - \mathfrak{X}}.$$

206. $X' = (Ax + BX + C) \div (ax + bX + c)$.

A differential equation bearing a resemblance to that of § 203 is

$$(ax + bX + c)X' = Ax + BX + C.$$

If the two constants c and C did not appear, then by dividing by x, each (x, X) function would be converted into one involving the ratio only of the two variables. But c and C can be got rid of by shifting parallelly the axes of co-ordinates from which x and X are measured, which change does not affect X'. If $\bar{x}$ and $\overline{X}$ be the new co-ordinates, then it is easily shown that the axes must be shifted so as to make

$$\bar{x} = x - \frac{Bc - bC}{Ab - aB} \text{ and } \overline{X} = X - \frac{Ca - cA}{Ab - aB}.$$

Then, since $\dfrac{d\overline{X}}{d\bar{x}} = \dfrac{dX}{dx} = X'$, by dividing out by $\bar{x}$, there results an equation of the form dealt with in § 203.

207. Particular case, $B = -a$.

If in the equation of last article $B = -a$, then the two terms with the common factor a combine to make the complete increment of xX. Thus the equation then reduces to

$$(bX + c)dX + a(Xdx + xdX) = (Ax + C)dx$$

the integration of which gives

$$\tfrac{1}{2}bX^2 + cX + axX - \tfrac{1}{2}Ax^2 - Cx + K = 0$$

in which K stands for the integration constant.

208. $X' + X\mathfrak{X}' = \Xi$.

Let $\mathfrak{X}'$ and Ξ be given functions of x, of which $\mathfrak{X}'$ is a function whose integral by dx can be directly found, namely $\mathfrak{X}$. Then, if the differential equation between X, X', and x be found to be

$$X' + X\mathfrak{X}' = \Xi;$$

this can be solved by the device of multiplying by what is called an "integrating factor," which means a factor which converts both sides of the equation into directly integrable functions. The factor which does this in the present case is e^{Ξ}, where Ξ is the integral of the given function Ξ' and e is the base of the natural logarithmic system. Since the x-gradient of e^{Ξ} is $e^{\Xi}\Xi'$, that of Xe^{Ξ} is $X'e^{\Xi} + X\Xi'e^{\Xi}$. Therefore, multiplying both sides of the differential equation by e^{Ξ} and integrating, there is obtained the integral equation

$$Xe^{\Xi} = \int \Xi e^{\Xi} dx + C.$$

This formula is of practical use only when Ξe^{Ξ} is a function which can be integrated either directly or by help of some transformation.

209. $X' + X\Xi' = X^n \Xi.$

This process may be followed in solving the differential equation

$$X' + X\Xi' = X^n \Xi$$

because $X^n\Xi$ is a function of x and may be inserted in place of Ξ in the above solution. Another solution, however, is obtained by dividing each side by X^n and multiplying by the integrating factor $(1-n)e^{(1-n)\Xi}$. The x-gradient of $X^r e^{s\Xi}$ is

$$(rX' + sX\Xi')X^{r-1}e^{s\Xi}$$

so that, taking $s = r$, the x-gradient of $X^r e^{r\Xi}$ is $(X' + X\Xi')X^{r-1} . re^{r\Xi}$. The first factor here is identical with the left-hand side of the differential equation of this article when each side is divided by X^n if $r = 1 - n$. Therefore the integration gives

$$X^{(1-n)}e^{(1-n)\Xi} = (1-n)\int \Xi e^{(1-n)\Xi} dx + C.$$

Provided $\Xi e^{(1-n)\Xi}$ be integrable, this formula will be of practical use.

210. $X = xF(X') + f(X').$

The following equation is generalised from that of § 202 by inserting the general function $F(X')$ of the x-gradient of X, in place of the special simple function X'.

$$X = xF(X') + f(X').$$

Taking the x-gradient of each side,

$$X' = F(X') + \{xF'(X') + f'(X')\}\frac{dX'}{dx}.$$

Transposing this,

$$\frac{dx}{dX'} - x\frac{F'(X')}{X' - F(X')} = \frac{f'(X')}{X' - F(X')}.$$

This is of the same form as that of § 208 with the substitutions

$$x \text{ in place of X}$$
$$\text{X}' \text{ ,, ,, ,, } x$$
$$-\frac{\text{F}'(\text{X}')}{\text{X}' - \text{F}(\text{X}')} \text{ ,, ,, ,, } \mathscr{X}'$$

and

$$\frac{f'(\text{X}')}{\text{X}' - \text{F}(\text{X}')} \text{ ,, ,, ,, } \Xi.$$

Therefore, if we use the shorthand symbol

$$\mathscr{X} \text{ for } -\int \frac{\text{F}'(\text{X}')}{\text{X}' - \text{F}(\text{X}')} d\text{X}'$$

so that $\mathscr{X}$ is a function of X' only; and further use

$$\Xi \text{ for } \frac{f'(\text{X}')}{\text{X}' - \text{F}(\text{X}')},$$

another function of X' only; the integration gives

$$xe^{\Xi} = \int \Xi e^{\Xi} d\text{X}' + \text{C};$$

which, if Ξe^{Ξ} be directly integrable by X', gives an algebraic equation between x and X'. Combining this with the original equation as simultaneous, the algebraic elimination of X' gives the desired integral equation involving only x and X.

211. General Equation of 1st Order of any Degree.—The preceding differential equations contain X' in the first power only. The general equation of the first order and of any degree may be expressed thus:—

$$\text{X}'^n + \mathscr{X}_{n-1}\text{X}'^{n-1} + \mathscr{X}_{n-2}\text{X}'^{n-2} + \cdots + \mathscr{X}_1\text{X}' + \mathscr{X}_0 = 0$$

where $\mathscr{X}_{n-1}, \mathscr{X}_{n-2}, \cdots \mathscr{X}_0$ are n different functions involving both x and X, and n is the **degree** of the equation. If possible this should first be solved for X' algebraically in terms of x and X. There will be n solutions giving n values of X' which may be symbolised by $\text{X}'_n, \text{X}'_{n-1}, \cdots \text{X}'_2, \text{X}'_1$, each of these values of X' being expressed in terms of x and X. This reduces the above equation of the nth degree to an equivalent series of n linear or first-degree equations; for instance the first of this series is

$$\text{X}' - \text{X}'_n = 0$$

where X'_n is a function, supposed now to be known, of x and X. Integrate each such linear equation if possible, by one of the methods already given. Let the integral solutions be here symbolised by

$$\phi_n(x, X, C) = 0 \; ; \quad \phi_{n-1}(x, X, C) = 0 \; ; \text{ etc., etc.}$$

Then the general solution, that is, the equation which includes all these various solutions, is either

$$\phi_n(x, X, C) \cdot \phi_{n-1}(x, X, C) \dashrightarrow \phi_2(x, X, C) \cdot \phi_1(x, X, C) = 0$$

or some other algebraically legitimate combination of these solutions.

212. Quadratic Equation of First Order.—Applying the result of last article to the equation of the second degree, namely,

$$X'^2 + X' \mathcal{X} + \Xi = 0 .$$

The algebraic solution of this for X' in terms of $\mathcal{X}$ and Ξ is

$$X' = -\frac{\mathcal{X}}{2} \pm \frac{1}{2} \sqrt{\mathcal{X}^2 - 4\Xi} .$$

The two integral solutions are, therefore,

and

$$\left. \begin{array}{c} X + \frac{1}{2}\int \mathcal{X}dx - \frac{1}{2}\int \sqrt{\mathcal{X}^2 - 4\Xi}\,dx + C = 0 \\[2mm] X + \frac{1}{2}\int \mathcal{X}dx + \frac{1}{2}\int \sqrt{\mathcal{X}^2 - 4\Xi}\,dx + C = 0 . \end{array} \right\}$$

As an easy example, take the differential equation

$$X'^2 + X' \sin x - \frac{\cos^2 x}{4} = 0 .$$

Therefore, $\sqrt{\mathcal{X}^2 - 4\Xi} = \sqrt{\sin^2 x + \cos^2 x} = 1$; and the two integral solutions are

and

$$\left. \begin{array}{c} X = \frac{1}{2}(\cos x - x) + C \\[2mm] X = \frac{1}{2}(\cos x + x) + C . \end{array} \right\}$$

213. Equation of Second Order with One Variable Absent.—In differential equations of the second order with only one independent variable, there may appear powers, trigonometrical, or any other kind of functions of all the four quantities

$$X'', X', X \text{ and } x.$$

I

Now X″ may be expressed in terms of X′ and either x or X by means of the substitutions

$$X'' \equiv \frac{dX}{dx} \qquad (a)$$

$$\equiv X'\frac{dX'}{dX} \qquad (b).$$

Therefore, if in any second-order differential equation X does not appear, it may be transformed by help of the substitution (a) so as to make X″ also disappear, leaving only

$$\frac{dX'}{dx}, \ X' \text{ and } x.$$

This is an equation of the first order between X′ and x, and may be solved by methods already explained so as to give X′ as a function of x $\left(i.e., \text{ so as to eliminate } \dfrac{dX'}{dx}\right)$. This again is a first-order equation between X and x, and by a second similar solution we may pass to the integral equation between X and x.

On the other hand, if x does not appear in any second-order equation, it may be reduced by the substitution (b), so that it will involve only

$$\frac{dX'}{dX}, \ X' \text{ and } X.$$

This is an equation of the first order again between X′ and X, whose solution gives an equation between X′ and X not involving $\dfrac{dX'}{dX}$; and from this again by a second integration, the desired integral relation between X and x may result.

214. Second Order Linear Equation.—The linear, or 1st degree, equation of the second order appears in a very general form as

$$X'' + X'f(x) + XF(x) = \phi(x),$$

where f, F and ϕ are any forms of function.

Provided this equation can be solved when $\phi(x) = 0$; then also, when $\phi(x)$ is any function, it may be reduced to an equation of the 1st order. Thus, let Ξ be a function of x which would be a solution if $\phi(x)$ were zero; that is, let

$$\Xi'' + \Xi'f(x) + \Xi F(x) = 0.$$

Give the name $\mathfrak{X}$ to the ratio of X, the true solution of the given equation, to Ξ; that is, let

$$X = \mathfrak{X}\Xi;$$

Then

$$X' = \mathfrak{X}'\Xi + \mathfrak{X}\Xi'$$

and

$$X'' = \mathfrak{X}''\Xi + 2\mathfrak{X}'\Xi' + \mathfrak{X}\Xi''.$$

Therefore, inserting these substitutions in the original equation, it becomes

$$\mathfrak{X}''\Xi + \mathfrak{X}'\{2\Xi' + \Xi f(x)\} + \mathfrak{X}\{\Xi'' + \Xi'f(x) + \Xi F(x)\} = \phi(x).$$

But the bracketed factor of the third term is zero; so that the transformed equation becomes

$$\mathfrak{X}''\Xi + \mathfrak{X}'\{2\Xi' + \Xi f(x)\} = \phi(x).$$

The supposition is that Ξ has been found; from which Ξ' can also be found in terms of x. This last form of the equation therefore contains only known functions of x besides $\mathfrak{X}''$ and $\mathfrak{X}'$. Now $\mathfrak{X}''$ is the first x-gradient of $\mathfrak{X}'$; and this is therefore a 1st order linear equation as between $\mathfrak{X}'$ and x. Thus, if any of the already explained, or any other, method of integrating 1st order linear equations be applicable, then $\mathfrak{X}'$ can be found as an explicit or implicit function of x, thus giving another 1st order equation between $\mathfrak{X}$ and x. By a second integration by one of these same methods, $\mathfrak{X}$ may then be found as a function of x; and finally $X = \mathfrak{X}\Xi$ can be obtained as the desired solution.

215. $X'' + aX' + bX = 0$.

The equation determining Ξ in § 214 is soluble or not according to the particular forms of the functions $f(\ )$ and $F'(\ )$: at any rate no reduction of the equation has yet been discovered showing, independently of the forms of $f(\ )$ and $F(\ )$, how it may be solved.

One simple case is that in which these functions are both constants. Let $f(x) \equiv a$ and $F(x) \equiv b$, a and b being both constants. The equation is then, using X instead of Ξ,

$$X'' + aX' + bX = 0.$$

Using the substitution (b) of § 213, this becomes

$$\left(a + \frac{dX'}{dX}\right)X' + bX = 0.$$

This may be written

$$\frac{d\mathrm{X}'}{d\mathrm{X}} = -\left(b\frac{\mathrm{X}}{\mathrm{X}'} + a\right)$$

which is soluble by the method of § 204; or it may be written

$$\mathrm{X}' = -\frac{b\mathrm{X}}{a + \dfrac{d\mathrm{X}'}{d\mathrm{X}}}$$

which is soluble by § 200.

By either method the solution is obtained which is printed in the *Classified Reference List*, at XI. 10.

216. $\mathrm{X}'' + a\mathrm{X}' + b\mathrm{X} = \phi(x)$.

In this simple case of $f(x) = a$ and $\mathrm{F}(x) = b$, the more general equation of § 214 becomes

$$\mathrm{X}'' + a\mathrm{X}' + b\mathrm{X} = \phi(x);$$

and its reduced form, when divided out by Ξ, becomes

$$\cancel{\mathscr{X}}'' + \cancel{\mathscr{X}}'\left\{2\frac{\Xi'}{\Xi} + b\right\} = \frac{\phi(x)}{\Xi}.$$

By § 215 both Ξ and Ξ' are known functions of x; and, therefore, this equation is of the form given in § 208, and can be integrated so as to give $\cancel{\mathscr{X}}'$, provided the function $\left\{2\dfrac{\Xi'}{\Xi} + b\right\}$ can be integrated by dx. This is integrable because, by reference to XI. 10 of the *Classified Reference List*, it is found that

$$2\frac{\Xi'}{\Xi} = -a + \sqrt{a^2 - 4b}\left\{\frac{1}{1 + \dfrac{\mathrm{B}}{\mathrm{A}}e^{-x\sqrt{a^2-4b}}} - \frac{1}{1 + \dfrac{\mathrm{A}}{\mathrm{B}}e^{x\sqrt{a^2-4b}}}\right\} \quad \text{when } a^2 > 4b$$

$$= -a - \sqrt{4b - a^2}\tan\left\{\frac{x}{2}\sqrt{4b - a^2} + \mathrm{C}'\right\} \qquad \text{when } a^2 < 4b$$

217. $\mathrm{X}^{(n)} = f(x)$.

If the nth x-gradient of X be called $\mathrm{X}^{(n)}$, and the process of repeating the integration of a function n times be symbolised by $\displaystyle\int^{(n)}$; then, if the differential equation of the nth order be

$$\mathrm{X}^{(n)} = f(x)$$

it has already been shown in § 154 that the integral equation between X and x is

$$\mathrm{X} = \int^{(n)} f(x)dx^n + \mathrm{C}_{n-1}x^{n-1} + \mathrm{C}_{n-2}x^{n-2} + \cdots + \mathrm{C}_1 x + \mathrm{C}_0.$$

218. $X^{(n)} = f(X) : X^{(n)} = kX$.

If the equation of the nth order be

$$X^{(n)} = f(X)$$

it is integrable only in particular cases. Thus in § 153 is given the case

$$X^{(n)} = kX$$

where k is any number $+$ or $-$. Let b be **any** number, and let T be the "modulus" of the system of logarithms of which the base is b. Take $\beta = Tk^{1/n}$. Then a solution of the above equation

$$X = b^{\beta x} \text{ or } \log_b X = \beta x .$$

If b be taken equal to e, the base of natural logarithms, then $T = 1$, and the solution is

$$\log_e X = k^{1/n} x .$$

If decimal logarithms be used, or $b = 10$; then $T = \cdot 434$, and

$$\log_{10} X = \cdot 434 k^{1/n} x .$$

Again in § 153 it is shown that if n be an **even** number, and if

$$X^{(n)} = (-1)^{n/2} kX$$

then an integral solution is

$$X = A \sin k^{1/n} x + B \cos k^{1/n} x$$

where A and B are constants of integration.*

219. $X'' = f(X)$.

When $n = 2$, this equation becomes

$$X'' = f(X) .$$

A general rule independent of the form of $f(\ )$ has been found for dealing with this second-order equation. Multiply each side by $2X'$. Then since $2X'X''$ is the x-gradient of X'^2, and since $X'dx = dX$, there results

$$X' = \left\{ 2 \int f(X) dX + A \right\}^{\frac{1}{2}} ;$$

from which, by another integration,

$$x + B = \int \frac{dX}{\left\{ 2 \int f(X) dX + A \right\}^{\frac{1}{2}}} .$$

---

* The $1/n^{\text{th}}$ root of k has n values. The insertion of these gives the n integration-constants of this n-th order equation.

As examples, the results of §§ 148 and 149 may be reproduced ; but these are included in the more general formulas of last article, § 218.

220. $X^{(n)} = f(X^{(n-1)})$.

If $X^{(n)}$ be found as a function of $X^{(n-1)}$; then, since the x-gradient of $X^{(n-1)}$ is $X^{(n)}$, if we call $X^{(n-1)}$ by the name $\mathcal{X}$, the equation may be written

$$\mathcal{X}' = f(\mathcal{X}),$$

the integration of which by dx gives

$$x + C = \int \frac{d\mathcal{X}}{f(\mathcal{X})},$$

that is, gives $X^{(n-1)}$ as a function of x, a case which has been already dealt with in § 217.

221. If $X^{(n)}$ be found as a function of $X^{(n-2)}$; then, calling $X^{(n-2)}$ by the name $\mathcal{X}$, we have $X^{(n)} = \mathcal{X}''$, and the equation becomes

$$\mathcal{X}'' = f(\mathcal{X}),$$

the integration of which, by § 219, gives $X^{(n-2)}$ as a function of x, and this reduces the integration to the case of § 217.

PART II.

CLASSIFIED REFERENCE TABLES

OF

INTEGRALS AND METHODS OF INTEGRATION,

IN ELEVEN SECTIONS.

TABLE OF CONTENTS.

NOTATION.

Letters near the beginning of the alphabet denote constants which may in general be positive or negative, whole or fractional, real quantities or numbers. Those near the end of the alphabet denote variables.

The symbol $\equiv$ stands for "denotes" or "is identical with."

The symbols $>$, $\not>$, $<$, $\not<$ stand for greater than, not greater than, less than, not less than, respectively.

The symbols $f(\ )$, $F(\ )$, $\phi(\ )$, $\psi(\ )$, denote any function of the quantity placed inside the brackets, except when restricted by the context.

X, Ξ, $\mathcal{X}$, χ are briefer symbols for functions of x.

L, M, N, P, Q, R also sometimes denote functions of the variable.

$Y \equiv$ any function of the variable y which is independent of x.

$f(a) \equiv$ the same function of a that $f(x)$ is of the variable x.

$$\left[f(x) \right]_a^b \equiv f(b) - f(a)$$

$$f'(x) \equiv \frac{df(x)}{dx} : \quad f''(x) \equiv \frac{d^2 f(x)}{dx^2} : \dots f^n(x) \equiv \frac{d^n f(x)}{dx^n}$$

$$X' \equiv \frac{dX}{dx} : \quad X'' \equiv \frac{d^2 X}{dx^2} : \dots X^{(n)} \equiv \frac{d^n X}{dx^n}$$

$$Y' \equiv \frac{dY}{dy} : \quad Y'' \equiv \frac{d^2 Y}{dy^2} : \dots Y^{(n)} \equiv \frac{d^n Y}{dy^n}$$

$$f'(X) \equiv \frac{df(X)}{dX} : \quad f'(Y) \equiv \frac{df(Y)}{dY}$$

$$f'_x(X) \equiv \frac{df(X)}{dx} : \quad f''_x(X) \equiv \frac{d^2 f(X)}{dx^2} : \dots f^n_x(X) \equiv \frac{d^n f(X)}{dx^n}$$

$\dfrac{\partial}{\partial x} \equiv$ partial differentiation with respect to x.

$f'_x(x,y) \equiv$ the differential coefficient with respect to x of the function $f(x,y)$ where x and y are independent variables.

$f''_{xy}(x,y) \equiv f''_{yx}(x,y) \equiv$ the second differential coefficient of $f(x,y)$ with respect to x and y, where x and y are independent variables.

$\int X dx \equiv$ Integral of X with respect to x

$$\int^{(2)} X dx^2 \equiv \int \int X dx^2 \equiv \int \int X dx dx \equiv \int \left\{ \int X dx \right\} dx$$

$$\int^{(n)} X dx^n \equiv \int \int -- (n \text{ symbols}) X dx^n \equiv \int \int -- (n \text{ symbols}) X dx dx -- (n dx's)$$

$$\int \int f(x,y) dy dx \equiv \int \left\{ \int f(x,y) dy \right\} dx$$

Σ or $\Sigma(\ ) \equiv$ Sum of a series of terms of the same type as that following Σ or placed within the brackets.

$$\int_a^b f(x) dx \equiv \left[\int f(x) dx \right]_a^b \equiv \text{the limiting value towards which}$$
$\left[\Sigma f(x) \delta x \right]_a^b$ approaches as δx approaches the limiting value 0.

$$\int_a f(x) dx \equiv \int_a^x f(x) dx$$

$n! \equiv 1 \times 2 \times 3 \cdots (n-1) \times n$; ($n$ being a positive integer).

$\log_{10} x \equiv \log x$ in the Decimal or Common system of logarithms.

$\log_e x \equiv \log x$ in the Natural or Neperian   „   „   .

$$e \equiv 2 \cdot 7182818 \cdots \equiv 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots \equiv \text{ base of Neperian logarithms}$$

$$exp(x) \equiv e^x \equiv 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$B_1, B_2 \cdots$ (Bernoulli's Numbers).*

In trigonometric functions the angles are given in " **circular measure**," the unit being the **radian** which $= \dfrac{180°}{\pi}$ or $57° \ 17' \ 45''$ nearly.

$\Gamma(n) \equiv \int_0^\infty e^{-x} x^{n-1} dx$ (the Gamma Function).   See X. 1–6.

ABBREVIATIONS.

Sp. Case $\equiv$ Special Case.

Exc. $\equiv$ Exceptional Case, or Case of failure.

* See *note* at end of Section VIII.

I. GENERAL THEOREMS.

1. $\int X'dx = X + C$

   $\int^{(2)} X''dx^2 = X + C_1 x + C_0$

   $\int^{(3)} X'''dx^3 = X + C_2 x^2 + C_1 x + C_0$

   $\int^{(n)} X^{(n)}dx^n = X + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} \cdots + C_1 x + C_0$

2. $\int_a^b f'(x)dx = f(b) - f(a)$

3. $\int af(x)dx = a\int f(x)dx$

4. $\int (X \pm \Xi \pm \mathcal{X} \cdots)dx = \int Xdx \pm \int \Xi dx \pm \int \mathcal{X}dx \cdots$

5. $\int_a^c Xdx = \int_a^b Xdx + \int_b^c Xdx$

   Sp. Case: $\int_a^b Xdx = -\int_b^a Xdx$

6. $\int_{x_1}^{x_2} f(x)dx = \int_{X_1}^{X_2} f\{\phi(X)\}\phi'(X)dX$

   where $x \equiv \phi(X)$ $\phi'(X) \equiv \dfrac{d\phi(X)}{dX} \equiv \dfrac{dx}{dX}$
   $x_1 \equiv \phi(X_1)$
   $x_2 \equiv \phi(X_2)$

   Sp. Case: $\int Xdx = \int \dfrac{X}{X'}dX$

7.   (INTEGRATION BY PARTS.)

   $\int X\Xi dx = X\int \Xi dx - \int \left\{ X'\int \Xi dx \right\} dx$

   or $\int X\Xi'dx = X\Xi - \int X'\Xi dx$

   Sp. Case: $\int Xdx = Xx - \int xdX$

8. $\int X\Xi dx = X\int\Xi dx - X'\int^{(2)}\Xi dx^2 + X''\int^{(3)}\Xi dx^3 + \cdots$

$$\pm X^{(n-1)}\int^{(n)}\Xi dx^n \mp \int\left\{X^{(n)}\int^{(n)}\Xi dx^n\right\}dx$$

Sp. Case: $\Xi = 1 : \int X dx = \frac{x}{1}X - \frac{x^2}{1\cdot 2}X' + \frac{x^3}{3!}X'' - \frac{x^4}{4!}X''' + \text{etc.}$

9. $\int\frac{X'}{X}dx = 2{\cdot}3026\cdots\log_{10}X + C = \log_e X + C$

10.   APPROXIMATE INTEGRATION.

(i.) $\displaystyle\int_a^b X dx = \frac{b-a}{2n}(X_0 + 2X_1 + 2X_2 + \cdots + 2X_{n-1} + X_n)$

      where X_0 is the value of X when $x = a$

       „ X_1 „   „   „   „ $= a + \dfrac{b-a}{n}$

       „ X_2 „   „   „   „ $= a + \dfrac{2(b-a)}{n}$

       „ X_r „   „   „   „ $= a + r.\dfrac{b-a}{n}$

(ii.)   (Simpson's Rule.)

$$\int_a^b X dx = \frac{b-a}{6n}\left\{\begin{array}{l}X_0 + X_{2n} + 4(X_1 + X_3 + X_5 + \cdots)\\ + 2(X_2 + X_4 + X_6 + \cdots + X_{2n-2})\end{array}\right\}$$

      where X_0, X_1, etc.$\equiv$same as in (i.).

(iii.) $\displaystyle\int_a^b f(x)dx = h\left[\tfrac{1}{2}\{f(b) + f(a)\} + f(a+h) + f(a+2h)\right.$

$$\left. + \cdots + f(b-h)\right]$$

$$+ \left[-\frac{B_1 h^2}{2!}f'(x) + \frac{B_2 h^4}{4!}f'''(x) - \frac{B_3 h^6}{6!}f^{(5)}(x) + \cdots\right]_a^b.*$$

* See *note* at end of Sect. VIII.

11.   METHOD OF UNDETERMINED COEFFICIENTS.

(i.)   If a function is known to be expressible in a certain form
        containing unknown coefficients, these coefficients can
        be determined by transforming the identity (*see below*)
        and equating coefficients of like terms on either side ;
        like terms being those whose variable part is the same
        function of the variable (*e.g.*, the same power, or the
        same trigonometrical function), or in both of which
        alike the variable is absent (constant terms).

        The transformation referred to may be :—
              (*a*) Differentiating both sides (as in III. B. 18).
              (*b*) Clearing of fractions (as in II. E. 2, etc.), or
                    otherwise.

(ii.)   Another method : Give to the primitive variable as many
        different values, in the identity, as there are coefficients
        to be determined ; whereby we get as many equations
        as are necessary to determine them.

GENERAL NOTE *re* IMAGINARIES.

Some of the formulas here given contain parts which would
become imaginary if the quantities involved took values outside
certain limits : becoming, *e.g.*, square roots and logarithms of
negative quantities, inverse sines of quantities greater than unity,
etc.   When a *definite* integral is deduced from the indefinite one,
these imaginaries, explicitly or implicitly, cancel one another, if
the subject of integration is itself real.   But, in many instances,
two or more forms are given for the integral of the same function
(*e.g.*, III. B. 6, VI. 5, 6), of which that one is to be selected
which, for the values of the constants in the particular problem
under consideration, is free from imaginaries.

Some of these formulas contain parts which are imaginary for
certain values of x only, whatever the constants may be, and
others do so for all values of x when the constants are outside
certain limits.   *E.g.*, the formula $\sin^{-1}\dfrac{x}{a}$ is imaginary when $x > a$,
but not when x lies between $-a$ and $+a$.   On the other hand, the
formula $\dfrac{1}{\sqrt{b}}\sinh^{-1}\left\{x\sqrt{\dfrac{b}{a}}\right\}$ contains imaginaries when b is
negative, whatever the value of x may be.

In these classified tables, the conditions under which a formula
involves imaginaries are, as a rule, pointed out in cases of the latter
sort, but not in those of the former.

Special Note as to Logarithmic Terms.

When a term of the form $A \log X$, where A is a constant, occurs in an integral, it becomes imaginary when x has such values as make X negative; but in such cases $A \log(-X)$, which is real, may always be used instead of $A \log X$, since it has the same differential coefficient as $A \log X$. This note applies to III. A. 3, 5, 11, 16, etc.

II.—CHIEF METHODS OF TRANSFORMATION.

A.   Express the subject of integration as the sum of a series of terms, and integrate these separately (see I. 4).   (Integration by decomposition or separation.)

$$E.g., \int \log\{(1+2x)(1+3x)\}dx \equiv \int \{\log(1+2x) + \log(1+3x)\}dx$$

$$\equiv \int \log(1+2x)dx + \int \log(1+3x)dx .$$

B.   Add and subtract the same quantity. $E.g.$,

$$\int \frac{xdx}{1+2x} = \int \frac{(x+\frac{1}{2})-\frac{1}{2}}{1+2x}\,dx$$

$$= \int \left\{ \frac{1}{2} - \frac{1}{2(1+2x)} \right\}dx .$$

C.   Multiply (or divide) numerator and denominator by the same quantity. $E.g.$,

$$\int \tan^3 x dx = \int \frac{\tan^3 x(1+\tan^2 x)dx}{1+\tan^2 x}$$

$$= \int \frac{\tan^3 x\, d\tan x}{1+\tan^2 x} .$$

Sp. Case :

$$\frac{M+N\sqrt{R}}{m+n\sqrt{R}} = \frac{(M+N\sqrt{R})(m-n\sqrt{R})}{m^2 - n^2 R}$$

D.   Expand in a series (see p. 146). $E.g.$,

$$\int \frac{dx}{\sqrt{(1-k^2\sin^2 x)}} = \int \left\{ 1 + \frac{1}{2}k^2\sin^2 x + \frac{1\cdot3}{2\cdot4}k^4\sin^4 x + \cdots \right\}dx .$$

Sp. Case : II. F.

E. Resolve rational fractions into partial fractions.   (See p. 146.)

F. Express a product of powers of sines and cosines as a sum of terms, each consisting of a sine or cosine multiplied by a constant. (See p. 147.)

K

G. Substitute $f(X)$ for x and $f'(X)dX$ for dx.   (See I. 6.)

$$\text{Sp. Case}:\quad \begin{aligned} f(X) &\equiv pX + q = x \\ dx &= pdX \end{aligned} \Bigg\} .$$

In the case of a definite integral, change the limits correspondingly (See I. 7), or else transform back to x after integration, before assigning limits.   (See p. 148.)

H. Differentiate or integrate an integral with respect to any quantity in it which is not a function of x, and a new integral is deduced.

K. Use integration by parts.   (See I. 7.)

II. D. *Chief Methods of Expansion in Series :—*

1. Binomial Theorem.
2. Exponential Theorem.
3. Expansion of $\log_e (1 \pm x)$.
4. Trigonometric series derived from the preceding expansions, by use of imaginaries.
5. Taylor's Theorem or Maclaurin's Theorem (including 1, 2, 3, as special cases).
6. Fourier's Expansion in series of sines and cosines.
7. Spherical Harmonics, Lamé's Functions, Bessel's Functions. Toroidal Functions, etc.

II. E. *Partial Fractions.*

$$\frac{F(x)}{f(x)} \equiv \frac{Ax^m + Bx^{m-1} + \cdots + H}{ax^m + bx^{m-1} + \cdots + h}$$

where m and n are positive integers.

1. If $m \not< n$ reduce $\dfrac{F(x)}{f(x)}$ by ordinary division to an integral function of x, + a fraction of similar form to the above, in which $m < n$.

2. *First Case.* $f(x) \equiv$ a product of simple factors, all different.

$$f(x) \equiv a(x - p)(x - q) \cdots (x - s)$$

$$\frac{F(x)}{f(x)} \equiv \frac{P}{x - p} + \frac{Q}{x - q} + \cdots + \frac{S}{x - s} .$$

To find P, Q ---- S, which are constants, use I. 11 (b).

Otherwise, $P = \dfrac{F(p)}{f'(p)}$, $Q = \dfrac{F(q)}{f'(q)}$ etc.

3. *Second Case.* Some factors repeated.

E.g. $f(x) \equiv a(x-p)^3(x-q)^2(x-r) \cdots$

$$\frac{F(x)}{f(x)} \equiv \frac{P_1}{x-p} + \frac{P_2}{(x-p)^2} + \frac{P_3}{(x-p)^3} + \frac{Q_1}{x-q} + \frac{Q_2}{(x-q)^2} + \frac{R}{x-r} + \cdots$$

To find $P_1 \, P_2 \cdots$ use I. 11. (b) *or* as in Williamson, § 40.

4. *Third Case.* $f(x)$ not expressible as a product of real simple factors. Avoid imaginaries by using quadratic factors; suppose them all different. *E.g.*

$$f(x) \equiv a(x^2 + kx + l)(x^2 + mx + n)(x-p)(x-q) \cdots$$
$$\frac{F(x)}{f(x)} \equiv \frac{Kx+L}{x^2+kx+l} + \frac{Mx+N}{x^2+mx+n} + \frac{P}{x-p} + \cdots$$

Determine the constants K, L, M, etc., by I. 11. (b). *Or* see Williamson, § 1.

5. *Fourth Case.* Some quadratic factors repeated. *E.g.*

$$f(x) \equiv a(x^2 + kx + l)^3(x^2 + mx + n)(x-p) \cdots$$
$$\frac{F(x)}{f(x)} \equiv \frac{K_1x+L_1}{x^2+kx+l} + \frac{K_2x+L_2}{(x^2+kx+l)^2} + \frac{K_3x+L_3}{(x^2+kx+l)^3}$$
$$+ \frac{Mx+N}{x^2+mx+n} + \frac{P}{x-p} + \cdots$$

Determine the constants $K_1 \, L_1$ etc., by I. 11. (b), *or* as in Williamson, § 43.

II. F. *Sines and Cosines.*

1. Use repeatedly
$$\mathrm{Sin}\, mx \cos nx \equiv \frac{1}{2}\left\{ \sin(m+n)x + \sin(m-n)x \right\}$$
$$\mathrm{Cos}\, mx \cos nx \equiv \frac{1}{2}\left\{ \cos(m+n)x + \cos(m-n)x \right\}$$
$$\mathrm{Sin}\, mx \sin nx \equiv \frac{1}{2}\left\{ \cos(m-n)x - \cos(m+n)x \right\}$$

E.g. $\mathrm{Sin}^3 x \cos 2x \cos x \equiv \dfrac{1}{8}\sin 4x - \dfrac{1}{16}\sin 6x - \dfrac{1}{16}\sin 2x$.

2. **Alternative method.**

$$\text{Use } X \equiv e^{ix} \therefore 2\cos x = X + X^{-1}$$
$$2i \sin x = X - X^{-1}$$
$$2\cos nx = X^n + X^{-n}$$
$$2i \sin nx = X^n - X^{-n}.$$

Express sines and cosines of x or its multiples in terms of X. Multiply out. Collect pairs of terms of the form C $(X^n \pm X^{-n})$, and reintroduce sines and cosines.

E.g. $\sin^3 x \cos 2x \cos x$

$$= \frac{1}{(2i)^3 2^2} (X - X^{-1})^3 (X^2 + X^{-2})(X + X^{-1})$$

$$= -\frac{1}{16} \frac{1}{2i} \{X^6 - X^{-6} - 2(X^4 - X^{-4}) + X^2 - X^{-2}\}$$

$$= -\frac{1}{16} \{\sin 6x - 2\sin 4x + \sin 2x\}.$$

II. G.—*Substitutions.*

1. $(ax + b)^n dx = \dfrac{1}{a} X^n dX$ $\left.\right\}$
2. $F(ax + b)dx = \dfrac{1}{a} F(X)dX$ $\left.\right\}$ $X \equiv ax + b$.

   Sp Case: $b = \pm \dfrac{\pi}{2}$. (See VI. head note.)

3. $\dfrac{dx}{x\sqrt{(ax^2 + b)}} = \dfrac{-dX}{\sqrt{(a + bX^2)}}$ $\qquad X \equiv \dfrac{1}{x}$.

4. $\dfrac{x^m dx}{(ax + b)^{q/p}} = \dfrac{p}{a^{m+1}}(X^p - b)^m X^{p-q-1} dX$

   $$X \equiv (ax + b)^{1/p}.$$

5. $F(ax^2 + bx + c)dx = F\{a(X^2 + k)\}dX$

   $$X \equiv x + \frac{b}{2a}, \quad k \equiv \frac{4ac - b^2}{4a^2}.$$

6. $\dfrac{dx}{\sqrt{\{(x - a)(x - b)\}}} = \dfrac{2dX}{1 - X^2}, \; X \equiv \sqrt{\left(\dfrac{x - a}{x - b}\right)}.$

7. $F(x^2 + k^2)dx = F(k^2\sec^2 X)k \sec^2 X dX$
$$X \equiv \tan^{-1}\frac{x}{k}.$$

Otherwise $= F(k^2\cosh^2 X)k \cosh X dX$
$$X \equiv \sinh^{-1}\frac{x}{k}.$$

8. $F\{(x^2 - k^2)^{\frac{1}{2}}\}dx = F(k \tan X)k \sec X \tan X dX$
$$X \equiv \sec^{-1}\frac{x}{k}.$$

Otherwise $= F(k \sinh X)k \sinh X . dX$
$$X \equiv \cosh^{-1}\frac{x}{k}.$$

9. $F\{(k^2 - x^2)^{\frac{1}{2}}\}dx = F(k \cos X)k \cos X dX$
$$X \equiv \sin^{-1}\frac{x}{k}.$$

10. $F(x, \log_e x)dx = F(e^X, X)e^X dX$
$$X \equiv \log_e x.$$

11. $F(b \cos x + c \sin x)dx = F\{\sqrt{(b^2 + c^2)} \sin X\}dX$
$$X \equiv x + \tan^{-1}\frac{b}{c}.$$

Otherwise, as in II. G. 12.

12. $F(a + b \cos x + c \sin x)dx = F\left\{\dfrac{a + b + 2cX + (a - b)X^2}{1 + X^2}\right\}\dfrac{2dX}{1 + X^2}$
$$X \equiv \tan\frac{x}{2}.$$

13. $F(a + b \cos^2 x + c \sin^2 x)dx = F\left\{\dfrac{a + b + (a + c)X^2}{1 + X^2}\right\}\dfrac{dx}{1 + X^2}$
$$X \equiv \tan x.$$

14. $F(\cos x, \sin^2 x) . \sin x dx = - F\{X, (1 - X^2)\}dX$
$$X \equiv \cos x.$$

15. $F(\sin x, \cos^2 x) . \cos x dx = F\{X, (1 - X^2)\}dX$
$$X \equiv \sin x.$$

16. $F(\sin^{-1}x)dx = F(X) \cos X dX,$
$$X \equiv \sin^{-1} x$$

and similarly for other inverse functions.

III.-IX.—TABLE OF INTEGRALS.

III. ALGEBRAIC FUNCTIONS.

III. A. *Mainly Rational.*

1. $\int a\,dx \quad = C + ax.$

2. $\int x^n dx \quad = C + \dfrac{x^{n+1}}{n+1}.$

    [Exc. $n = -1$. (See III. A. 3.)]

3. $\int \dfrac{dx}{x} \quad = C + 2{\cdot}302585\cdots \times \log_{10}x$

             $= C + \log_e x.$

4. $\int (ax+b)^n = C + \dfrac{1}{(n+1)a}(ax+b)^{n+1}.$

    Exc. $n = -1$. (See III. A. 5.)

5. $\int \dfrac{dx}{ax+b} \quad = C + \dfrac{2{\cdot}3026\cdots}{a} \times \log_{10}(ax+b)$

              $= C + \dfrac{1}{a}\log_e(ax+b).$

6. $\int \dfrac{Ax+B}{ax+b}dx = C + \dfrac{A}{a}x + \dfrac{aB-Ab}{a^2}\log_e(ax+b).$

7. $\int x^m(ax+b)^n dx.$ Use II. A., or IX. A. 1, or III. A. 20.

8. $\int \dfrac{dx}{1+x^2} \quad = C + \tan^{-1}x = C - \cot^{-1}x.$

9. $\int \dfrac{dx}{1-x^2} \quad = C + \tfrac{1}{2}\log_e\dfrac{1+x}{1-x} \quad [x<1]$

              $= C + \tfrac{1}{2}\log_e\dfrac{x+1}{x-1} \quad [x>1].$

10. $\int \dfrac{dx}{ax^2+b} \quad = \dfrac{1}{\sqrt{(ab)}}\int \dfrac{dX}{1+X^2}$ where $X \equiv x\sqrt{\dfrac{a}{b}}$, when $ab>0$.

          (See III. A. 8.)

        $= \dfrac{1}{\sqrt{(-ab)}}\int \dfrac{dX}{1-X^2}$ where $X \equiv x\sqrt{\left(-\dfrac{a}{b}\right)}$, when $ab<0$.

          (See III. A. 9.)

11. $\int \dfrac{dx}{(Ax+B)(ax+b)} = C + \dfrac{1}{Ab-aB}\log_e\left(\dfrac{Ax+B}{ax+b}\right).$

12. $\int \dfrac{x^m dx}{1+x^n}$ where m is a positive integer or 0, and n a positive integer. If $m \not< n$, use II. E. 1; if $m < n$, thus:—

$= C + \dfrac{(-1)^m}{n}\log_e(1+x) - \dfrac{1}{n}\Sigma\left\{\cos\dfrac{r(m+1)\pi}{n}\log_e(x^2-2x\cos\dfrac{r\pi}{n}+1)\right\}$

$\quad + \dfrac{2}{n}\Sigma\left\{\sin\dfrac{r(m+1)\pi}{n}\tan^{-1}\left(\dfrac{x-\cos\dfrac{r\pi}{n}}{\sin\dfrac{r\pi}{n}}\right)\right\}.$

    N.B.—If n is odd, r takes the values 1, 3, 5 $\cdots$ $(n-2)$.
       If n is even, r ,,   ,,   1, 3, 5 $\cdots$ $(n-1)$, and the
       term $\dfrac{(-1)^m}{n}\log_e(1+x)$ is *omitted*.

13. $\int \dfrac{x^m dx}{1-x^n}$ where m is a positive integer or 0, and n is an odd positive integer.

    $= (-1)^{m+1}\int\dfrac{X^m dX}{1+X^n}$, where $X \equiv -x$.   (See III. A. 12.)

14. $\int \dfrac{x^m dx}{1-x^n}$ where m is a positive integer or 0, and n is an even positive integer.

    $= C + \dfrac{1}{n}\log_e(1-x) - \dfrac{(-1)^m}{n}\log_e(1+x)$

    $\quad - \dfrac{1}{n}\Sigma\left\{\cos\dfrac{r(m+1)\pi}{n}\log_e(x^2-2x\cos\dfrac{r\pi}{n}+1)\right\}$

    $\quad + \dfrac{2}{n}\Sigma\left\{\sin\dfrac{r(m+1)\pi}{n}\tan^{-1}\left(\dfrac{x-\cos\dfrac{r\pi}{n}}{\sin\dfrac{r\pi}{n}}\right)\right\}$

    where r takes the values 2, 4, 6 $\cdots$ $(n-2)$.

15. $\int \dfrac{x^m dx}{ax^n+b} = \dfrac{1}{b}\left(\dfrac{b}{a}\right)^{(m+1)/n}\int\dfrac{X^m dX}{1+X^n}$ where $X \equiv x\left(\dfrac{a}{b}\right)^{1/n}$, if $ab < 0$.
    See III. A. 12, if m and n are positive integers.

    $= \dfrac{1}{b}\left(-\dfrac{b}{a}\right)^{(m+1)/n}\int\dfrac{X^m dX}{1-X^n}$ where $X \equiv x\left(-\dfrac{a}{b}\right)^{1/n}$, if $ab < 0$.
    See III. A. 13, 14, if m and n are positive integers.
             Otherwise, see IX. A. I.

16. $\int \dfrac{dx}{ax^2 + bx + c} = C + \dfrac{2}{\sqrt{(-A)}} \tan^{-1}\dfrac{2ax+b}{\sqrt{(-A)}}$ where $A \equiv b^2 - 4ac,$
$\hspace{10cm}$ if $A < 0$

$\hspace{3cm} = C + \dfrac{1}{\sqrt{A}}\log_e\dfrac{2ax+b-\sqrt{A}}{2ax+b+\sqrt{A}} \hspace{2cm}$ if $A > 0$.

17. $\int \dfrac{(Ax+B)dx}{ax^2+bx+c} = \dfrac{A}{2a}\log_e(ax^2+bx+c) + \dfrac{2aB-Ab}{2a}\int\dfrac{dx}{ax^2+bx+c}.$

$\hspace{4cm}$ (See III. A. 16.)

18. $\int \dfrac{Ax+B}{X^n}dx$ where $X \equiv ax^2 + bx + c$

$\hspace{2cm} = \dfrac{-A}{2(n-1)aX^{n-1}} + \dfrac{2Ba-Ab}{2a}\int\dfrac{dx}{X^n}.$ (See IX. A. 4.)

$\hspace{3cm}$ Sp. Case: $A = 0.$ (See IX. A. 4.)

19. $\int \dfrac{Ax^m + Bx^{m-1} + \cdots + K}{ax^n + bx^{n-1} + \cdots + p}dx,$ where m, n are positive integers

Reduce by II. E. to terms like these :—

$\int Ax^q dx, \int\dfrac{Adx}{x-p}, \int\dfrac{Adx}{(x-p)^r}, \int\dfrac{Lx+M}{x^2+lx+m}dx, \int\dfrac{Lx+M}{(x^2+lx+m)^r}dx \;;$

for which, see III. A. 2, 5, 4, 17, 18.

20. $\int x^m(ax^n+b)^{p/q}dx = \dfrac{q}{na}\int X^{p+q-1}\left(\dfrac{X^q-b}{a}\right)^{(m+1)/n-1}dX$

$\hspace{4cm}$ where $X \equiv (ax^n+b)^{1/q}$

$\hspace{0.5cm} = -\dfrac{q}{n}\int b^{(m+1)/n+p/q}(X^q-a)^{-(m+1)/n-p/q-1}X^{p+q-1}dX$

$\hspace{4cm}$ where $X \equiv (bx^{-n}+a)^{1/q}.$

Use the former when $\dfrac{m+1}{n}$ is a positive integer.

$\hspace{1cm}$ „ $\hspace{0.7cm}$ latter $\hspace{0.7cm}$ „ $\hspace{0.7cm} \dfrac{m+1}{n} + \dfrac{p}{q}$ is a negative integer.

In either case, expand the binomial factor and use II. A. and III. A. 2.

Definite Integrals with Numerical (or Particular) Limits.

21. $\displaystyle\int_0^\infty \frac{x^m dx}{1+x^n} = \frac{\pi}{n \sin\dfrac{(m+1)\pi}{n}}$ where n and m are even positive integers, and $m < n$.

22. $\displaystyle\int_0^1 \frac{dx}{\sqrt[n]{(1-x^n)}} = \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}$ if $n > 1$.

23. $\displaystyle\int_0^1 \frac{dx}{\sqrt[n]{(1+x^n)}} = \frac{\pi}{n}\cot\frac{\pi}{n}$ „ „

24. $\displaystyle\int_0^1 \frac{(x^m + x^n)dx}{(1+x)^{m+n+2}} = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$.   (See X. 1–6.)

25. $\displaystyle\int_0^1 x^m(1-x)^n dx = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$.   „   „

26. $\displaystyle\int_0^\infty \frac{x^m dx}{(1+x)^{m+n+2}} =$ „   „   „

27. $\displaystyle\int_0^a x^m(a-x)^n dx = a^{m+n+1}\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$ „   „

III. B. *Quadratic Surds.*

1. $\displaystyle\int (ax+b)^{\frac{1}{2}}dx = C + \frac{2}{3a}(ax+b)^{\frac{3}{2}}$.

2. $\displaystyle\int f\{(ax+b)^{\frac{1}{2}}\}dx = \frac{2}{a}\int Xf(X)d(X)$
   where $X \equiv (ax+b)^{\frac{1}{2}}$.

Sp. Cases :—

(1) $\displaystyle\int x(ax+b)^{\frac{1}{2}}dx = C + \frac{1}{5a^2}(ax+b)^{\frac{5}{2}} - \frac{b}{3a^2}(ax+b)^{\frac{3}{2}}$.

(2) $\displaystyle\int \frac{dx}{x(ax+b)^{\frac{1}{2}}} = 2\int \frac{dX}{X^2 - b}$.   (See III. A. 10.)

3. $\displaystyle\int \frac{dx}{(x^2+1)^{\frac{1}{2}}} = C + \log_e\{x + (x^2+1)^{\frac{1}{2}}\} = C + \sinh^{-1}x$.

4. $\displaystyle\int \frac{dx}{(x^2-1)^{\frac{1}{2}}} = C + \log_e\{x + (x^2-1)^{\frac{1}{2}}\} = C + \cosh^{-1}x$.

5. $\displaystyle\int \frac{dx}{(1-x^2)^{\frac{1}{2}}} = C + \sin^{-1}x = C - \cos^{-1}x$.

6. $\displaystyle\int \frac{dx}{(ax^2+b)^{\frac{1}{2}}} = C + \frac{1}{\sqrt{a}}\log_e\{x\sqrt{a} + (ax^2+b)^{\frac{1}{2}}\}$ if $a>0$ and $b>0$

$$= C + \frac{1}{\sqrt{a}}\sinh^{-1}x\sqrt{\frac{a}{b}} \qquad \text{,, } a>0 \text{ ,, } b>0$$

$$= C + \frac{1}{\sqrt{a}}\cosh^{-1}x\sqrt{\frac{-a}{b}} \qquad \text{,, } a>0 \text{ ,, } b<0$$

$$= C + \frac{1}{\sqrt{(-a)}}\sin^{-1}x\sqrt{-\frac{a}{b}} \qquad \text{,, } a<0 \text{ ,, } b>0$$

Otherwise : put $x \equiv X \sqrt{\dfrac{b}{a}}$ or $x \equiv X \sqrt{\left(-\dfrac{b}{a}\right)}$, and use III. B. 3, 4, or 5.

7. $\displaystyle\int \frac{x\,dx}{(ax^2+b)^{\frac{1}{2}}} = C + \frac{1}{a}(ax^2+b)^{\frac{1}{2}}$.

8. $\displaystyle\int \frac{dx}{x(ax^2+b)^{\frac{1}{2}}} = -\int \frac{dX}{(a+bX^2)^{\frac{1}{2}}}$ where $X \equiv \dfrac{1}{x}$. (See III. B. 6.)

9. $\displaystyle\int (Ax+B)(ax^2+b)^{\frac{1}{2}}dx = C + \left(\frac{A}{3}x^3 + \frac{B}{2}x + \frac{Ab}{3a}\right)(ax^2+b)^{\frac{1}{2}}$

$$+ \frac{Bb}{2}\int \frac{dx}{(ax^2+b)^{\frac{1}{2}}}. \quad \text{(See III. B. 6.)}$$

Sp. Case : $\displaystyle\int x(ax^2+b)^{\frac{1}{2}}dx = C + \frac{1}{3a}(ax^2+b)^{\frac{3}{2}}$.

10. $\displaystyle\int \frac{dx}{(2ax-x^2)^{\frac{1}{2}}} = C + \text{vers}^{-1}\frac{x}{a} = C + \cos^{-1}\frac{a-x}{a}$.

11. $\displaystyle\int \frac{dx}{(2ux+x^2)^{\frac{1}{2}}} = C + \cosh^{-1}\frac{a+x}{a} = C + \log_e\{a+x + (2ax+x^2)^{\frac{1}{2}}\}$.

12. $\displaystyle\int \frac{dx}{x(2ax \pm x^2)^{\frac{1}{2}}} = C - \frac{(2ax \pm x^2)^{\frac{1}{2}}}{ax}$.

13. $\displaystyle\int \frac{dx}{(ax^2+bx+c)^{\frac{1}{2}}} = 2\sqrt{a}\int \frac{dX}{(4a^2X^2 + 4ac - b^2)^{\frac{1}{2}}}$.

Where $X \equiv x + \dfrac{b}{2a}$. (See III. B. 6.)

14. $\int \dfrac{(Ax+B)dx}{(ax^2+bx+c)^{\frac{3}{2}}} = \dfrac{A}{a}(ax^2+bx+c)^{\frac{1}{2}} + \dfrac{2Ba-Ab}{2a}\int \dfrac{dx}{(ax^2+bx+c)^{\frac{1}{2}}}\,.$

<div align="center">(See II. B. 13.)</div>

15. $\int (Ax+B)(ax^2+bx+c)^{\frac{1}{2}}dx$

$$= C + \left\{\frac{A}{3}x^2 + \left(\frac{B}{2}+\frac{Ab}{12a}\right)x + \frac{Bb}{4a} + \frac{Ac}{3a} - \frac{Ab^2}{8a^2}\right\}(ax^2+bx+c)^{\frac{1}{2}}$$

$$+ \left(\frac{Bc}{2} - \frac{Abc}{4a} - \frac{Bb^2}{8a} + \frac{Ab^3}{16a^2}\right)\!\int \frac{dx}{(ax^2+bx+c)^{\frac{1}{2}}}\,.$$

<div align="center">(See III. B. 13.)</div>

16. $\int \dfrac{(Ax+B)dx}{(ax^2+bx+c)^{\frac{3}{2}}} = C + 2\,\dfrac{bB - 2cA + (2aB-bA)x}{(4ac-b^2)(ax^2+bx+c)^{\frac{1}{2}}}\,.$

17. $\int \dfrac{Ax^{2n-1}+Bx^{2n-2}+\cdots+K}{(ax^2+bx+c)^{n+\frac{1}{2}}}dx = C + \dfrac{Lx^{2n-1}+Mx^{2n-2}+\cdots+R}{(ax^2+bx+c)^{n-\frac{1}{2}}}$

if n is a positive integer; where L, M $\cdots\cdot$ R are constants to be determined by I. 11 (a).

18. $\int \dfrac{Ax^m+Bx^{m-1}+\cdots+K}{(ax^2+bx+c)^{\frac{1}{2}}}dx$

$$= (Px^{m-1}+Qx^{m-2}+\cdots+S)(ax^2+bx+c)^{\frac{1}{2}} + \int \frac{Mdx}{(ax^2+bx+c)^{\frac{1}{2}}}$$

where the constants P, Q, $\cdots\cdot$ S, M are determined by I. 11 (a). For the last integral, see III. B. 13. Otherwise, see IX. A. 3.

IV. LOGARITHMS AND EXPONENTIALS.

1. $\int e^x dx \quad = C + e^x$ ($e \equiv$ base of Neperian Logarithms).

2. $\int a^{nx} dx \quad = C + \dfrac{1}{n \log_e a} a^{nx}$.

    Sp. Case: $\int e^{nx} dx = C + \dfrac{1}{n} e^{nx}$.

3. $\int \log_e x \, dx \quad = C + x \log_e x - x$.

4. $\int \log_b x \, dx \quad = C + x(\log_b x - \log_b e)$.

    Sp. Case: $b = 10$, $\int \log_{10} x \, dx = C + x(\log_{10} x - \cdot 43429 \cdots)$.

5. $\int (\log_e x)^n dx = C + x\{(\log_e x)^n - n(\log_e x)^{n-1} + n(n-1)(\log_e x)^{n-2}$
$- \cdots \pm n!\}$.

6. $\int x^m e^x dx \quad = C + e^x\{x^m - mx^{m-1} + m(m-1)x^{m-2} \cdots - \cdots \pm m!\}$.

7. $\int a^{\log_b x} dx \quad = \int x^{\log_b a} dx$. (See III. A. 2).

For other formulæ involving logarithms or exponentials, see VIII. 7–25 and IX. C.

Definite Integrals with Numerical (or Particular) Limits.

8. $\int_0^\infty e^{-ax^2} dx \quad = \dfrac{1}{2} \sqrt{\dfrac{\pi}{a}}$ where $a > 0$.

9. $\int_0^1 \left(\log \dfrac{1}{x}\right)^n dx = \int_0^\infty e^{-x} x^n dx = \Gamma(n+1)$. (See X. 1–6.)

V.—HYPERBOLIC FUNCTIONS.

$$\left[\begin{array}{l} \text{Sinh}x \equiv \tfrac{1}{2}(e^x - e^{-x}); \quad \cosh x \equiv \tfrac{1}{2}(e^x + e^{-x}); \\[4pt] \tanh x \equiv \dfrac{\sinh x}{\cosh x}; \quad \coth x \equiv \dfrac{1}{\tanh x}; \\[8pt] \text{sech}x \equiv \dfrac{1}{\cosh x}; \quad \text{cosech}x \equiv \dfrac{1}{\sinh x}; \\[8pt] gdx = \cos^{-1}\text{sech}x = \sin^{-1}\tanh x = \tan^{-1}\sinh x = 2\tan^{-1}\tanh\dfrac{x}{2}. \end{array} \right]$$

1. $\displaystyle\int \sinh x . dx \quad = C + \cosh x .$

2. $\displaystyle\int \cosh x . dx \quad = C + \sinh x .$

3. $\displaystyle\int \tanh x . dx \quad = C + \log_e \cosh x .$

4. $\displaystyle\int \coth x . dx \quad = C + \log_e \sinh x .$

5. $\displaystyle\int \text{sech}x . dx \quad = C + 2\tan^{-1}e^x = C + gdx .$

6. $\displaystyle\int \text{cosech}x . dx = C + \log_e \dfrac{e^x - 1}{e^x + 1} = C + \log_e \tanh\dfrac{x}{2} .$

Note.—Every formula in VI. gives rise to a corresponding formula in hyperbolic functions, and *vice versâ*, by writing ix for x and using the identities :—

$$\sin ix = i \sinh x$$
$$\cos ix = \cosh x$$
$$\tan ix = i \tanh x$$

where $i \equiv \sqrt{(-1)}$.

VI.—TRIGONOMETRICAL FORMS.

Note.—The substitution of $px+q$ for x, and $p.dx$ for dx, gives more general forms of VI., 16, 19-25.

Sp. Case: $-x+\dfrac{\pi}{2}$ for x and $-dx$ for dx changes every trigonometrical ratio of x into its complementary function.

1. $\displaystyle\int \sin(px+q)dx = C - \frac{1}{p}\cos(px+q)$.

   Sp. Case: $\displaystyle\int \sin x\,dx = C - \cos x$.

2. $\displaystyle\int \cos(px+q)dx = C + \frac{1}{p}\sin(px+q)$.

   Sp. Case: $\displaystyle\int \cos x\,dx = C + \sin x$.

3. $\displaystyle\int \tan(px+q)dx = C - \frac{1}{p}\log_e \cos(px+q)$.

   Sp. Case: $\displaystyle\int \tan x\,dx = C - \log_e \cos x$.

4. $\displaystyle\int \cot(px+q)dx = C + \frac{1}{p}\log_e \sin(px+q)$.

   Sp. Case: $\displaystyle\int \cot x\,dx = C + \log_e \sin x$.

5. $\displaystyle\int \sec(px+q)dx = C + \frac{1}{2p}\log_e \frac{1+\sin(px+q)}{1-\sin(px+q)}$

   $\qquad\qquad = C + \frac{1}{p}\log_e \tan\left(\frac{\pi}{4}+\frac{px+q}{2}\right)$.

6. $\displaystyle\int \operatorname{cosec}(px+q)dx = C + \frac{1}{2p}\log_e \frac{1-\cos(px+q)}{1+\cos(px+q)}$

   $\qquad\qquad = C + \frac{1}{p}\log_e \tan\frac{px+q}{2}$.

7. $\displaystyle\int \sin^2(px+q)dx = C + \frac{x}{2} - \frac{1}{2p}\sin(px+q)\cos(px+q)$.

8. $\displaystyle\int \cos^2(px+q)dx = C + \frac{x}{2} + \frac{1}{2p}\sin(px+q)\cos(px+q)$.

9. $\int \tan^2(px+q)dx \quad = C - x + \dfrac{1}{p}\tan(px+q).$

10. $\int \cot^2(px+q)dx \quad = C - x - \dfrac{1}{p}\cot(px+q).$

11. $\int \sec^2(px+q)dx \quad = C + \dfrac{1}{p}\tan(px+q).$

12. $\int \operatorname{cosec}^2(px+q)dx = C - \dfrac{1}{p}\cot(px+q).$

13. $\int \dfrac{\sin(px+q)dx}{\cos^2(px+q)} = C + \dfrac{1}{p}\sec(px+q).$

14. $\int \dfrac{\cos(px+q)dx}{\sin^2(px+q)} = C - \dfrac{1}{p}\operatorname{cosec}(px+q).$

15. $\int \dfrac{dx}{\sin(px+q)\cos(px+q)} = C + \dfrac{1}{p}\log_e \tan(px+q).$

16. $\int \sin^n x\,dx, \quad \int \cos^n x\,dx, \quad \int \dfrac{dx}{\sin^n x}, \quad \int \dfrac{dx}{\cos^n x}, \quad \int \tan^n x\,dx, \quad \int \cot^n x\,dx.$

$\left\{\begin{array}{l}\text{For the first two integrals use II. F. or IX. B. 1, 2.} \\ \text{,, \quad ,, \quad second pair \quad ,, \quad IX. B. 3, 4.} \\ \text{,, \quad ,, \quad last pair \quad\quad ,, \quad VI. 21, 22.}\end{array}\right.$

See also VI. 19.

17. $\int \sin(px+q)\cos^n(px+q)dx = C - \dfrac{1}{(n+1)p}\cos^{n+1}(px+q).$

18. $\int \cos(px+q)\sin^n(px+q)dx = C + \dfrac{1}{(n+1)p}\sin^{n+1}(px+q).$

19. $\int \sin^m x \cos^n x\,dx.$ Four methods.

Method I. (1) if m is an odd positive integer, use $X \equiv \cos x$
    (2) ,, n ,, ,, ,, ,, $X \equiv \sin x$
    (3) ,, $m+n$,, even negative ,, ,, $X \equiv \tan x$

and the integrals become rational.

Method II. If m and n are positive integers, use II. F.

Method III. Use IX. B. 5, 6, 7 or 8.

Method IV. If m or n or both are fractional, use $X \equiv \sin x$ or $\equiv \cos x$, and expand the binomial factor which results.

20. $\int \sin^m x \cos^n x \sin qx \cos rx \cdots\cdot dx$.

   Where m, n, q, r ---- are positive integers, use II. F.

21. $\int \tan^n x\, dx$.

   If n is even, $= C + \dfrac{\tan^{n-1}x}{n-1} - \dfrac{\tan^{n-3}x}{n-3} + \cdots \pm \tan x \mp x$.

   „ „ odd $= C + \dfrac{\tan^{n-1}x}{n-1} - \dfrac{\tan^{n-3}x}{n-3} + \cdots \pm \dfrac{\tan^2 x}{2} \pm \log_e \cos x$.

22. $\int \cot^n x\, dx = C - \dfrac{1}{n-1} \cot^{n-1}x + \dfrac{1}{n-3} \cot^{n-3}x \cdots\cdot \pm \cot x \mp x$,
$$\text{if } n \text{ even,}$$
$$= C - \frac{1}{n-1} \cot^{n-1}x + \frac{1}{n-3} \cot^{n-3}x \cdots\cdots \pm \tfrac{1}{2} \cot^2 x$$
$$\pm \log_e \sin x, \text{ if } n \text{ odd.}$$

23. $\int \dfrac{dx}{a + b \cos x} = C + \dfrac{1}{\sqrt{(b^2 - a^2)}} \cosh^{-1} \dfrac{a \cos x + b}{a + b \cos x}$ if $b^2 > a^2$
$$= C + \frac{1}{\sqrt{(a^2 - b^2)}} \cos^{-1} \frac{a \cos x + b}{a + b \cos x} \quad \text{if } b^2 < a^2.$$

24. $\int \dfrac{dx}{a + b \cos x + c \sin x} = \int \dfrac{dX}{a + \sqrt{(b^2 + c^2)} \cos X}$

   where $X \equiv x - \tan^{-1} \dfrac{c}{b}$.   (See VI. 23.)

   Sp. Case :   (1) when $c = 0$, it becomes VI. 23.

            (2) when $b = 0$,   ,,   ,, $\int \dfrac{dx}{a + c \sin x}$.

25. $\int \dfrac{\cos^m x\, dx}{(a + b \cos x)^n} = \dfrac{1}{(a^2 - b^2)^{n-\frac{1}{2}}} \int (a \cos X - b)^m (a - b \cos X)^{n-m-1} dX$,

      if $a > b$, $m + 1 < n$, where $\tan \dfrac{X}{2} \equiv \sqrt{\dfrac{a-b}{a+b}} \tan \dfrac{x}{2}$.

$$= \frac{1}{(b^2 - a^2)^{n-\frac{1}{2}}} \int (b - a \cosh X)^m (b \cosh X - a)^{n-m+1} dX,$$

      if $a < b$, $m + 1 < n$, where $\tanh \dfrac{X}{2} \equiv \sqrt{\dfrac{b-a}{b+a}} \cdot \tan \dfrac{x}{2}$.

Expand in each case by Binomial Theorem, and use II. A., then VI. 16 or 19, or Note at end of V.

Definite Integrals with Particular [or Numerical] Limits.

26. $\displaystyle\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \dfrac{\Gamma(\tfrac{1}{2}).\Gamma\!\left(\dfrac{n+1}{2}\right)}{2\Gamma\!\left(\dfrac{n}{2}+1\right)}.$ (See X. 1–6.)

27. $\displaystyle\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \dfrac{\Gamma\!\left(\dfrac{m+1}{2}\right).\Gamma\!\left(\dfrac{n+1}{2}\right)}{2\Gamma\!\left(\dfrac{m+n+2}{2}\right)}.$,,   ,,

VII.—INVERSE FUNCTIONS.

[*Note.*—These can be transformed into integrals involving the corresponding **direct** functions by substitutions like $X \equiv \sin^{-1}x$ $\therefore x \equiv \sin X \therefore dx \equiv \cos X dX$, etc.]

N.B.—$\sinh^{-1}x = \log_e\{x + \sqrt{(1 + x^2)}\}$.

$$\cosh^{-1}x = \log_e\{x \pm \sqrt{(x^2 - 1)}\} ; \ x > 1.$$

$$\tanh^{-1}x = \frac{1}{2}\log_e\frac{1 + x}{1 - x} ; \ x < 1.$$

$$\coth^{-1}x = \frac{1}{2}\log_e\frac{x + 1}{x - 1} ; \ x > 1.$$

$$\text{sech}^{-1}x = \log_e\frac{1 \pm \sqrt{(1 - x^2)}}{x} ; \ x < 1.$$

$$\text{cosech}^{-1}x = \log_e\frac{1 + \sqrt{(1 + x^2)}}{x}, \text{ if } x > 0$$

$$= \log_e\frac{1 - \sqrt{(1 + x^2)}}{x}, \text{ if } x < 0.$$

$$\text{gd}^{-1}x = \text{sech}^{-1}\cos x = \tanh^{-1}\sin x = \sinh^{-1}\tan x$$

$$= 2\tanh^{-1}\tan\frac{x}{2} = \log_e\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{1}{2}\log_e\frac{1 + \sin x}{1 - \sin x}.$$

---

1. $\int \sin^{-1}x\,dx \quad = C + x\sin^{-1}x \pm (1 - x^2)^{\frac{1}{2}}$.

2. $\int \cos^{-1}x\,dx \quad = C + x\cos^{-1}x \mp (1 - x^2)^{\frac{1}{2}}$

$$= \int\left(\frac{\pi}{2} - \sin^{-1}x\right)dx.$$

3. $\int \tan^{-1}x\,dx \quad = C + x\tan^{-1}x - \frac{1}{2}\log_e(1 + x^2)$.

4. $\int \cot^{-1}x\,dx \quad = C + x\cot^{-1}x + \frac{1}{2}\log_e(1 + x^2)$.

5. $\int \sec^{-1}x\,dx \quad = C + x\sec^{-1}x - \log_e\{x + \sqrt{(x^2 - 1)}\}$

$$= C + x\sec^{-1}x - \cosh^{-1}x.$$

6. $\int \operatorname{cosec}^{-1}x\,dx = C + x\,\operatorname{cosec}^{-1}x + \log_e\{x + \sqrt{(x^2 - 1)}\}$
$$= C + x\,\operatorname{cosec}^{-1}x + \cosh^{-1}x.$$

7. $\int \sinh^{-1}x\,dx = C + x\,\sinh^{-1}x - \sqrt{(1 + x^2)}.$

8. $\int \cosh^{-1}x\,dx = C + x\,\cosh^{-1}x - \sqrt{(x^2 - 1)}.$

9. $\int \tanh^{-1}x\,dx = C + x\,\tanh^{-1}x + \dfrac{1}{2}\log_e(1 - x^2).$
$$= C + \frac{1+x}{2}\log_e(1 + x) + \frac{1-x}{2}\log_e(1 - x).$$

10. $\int \coth^{-1}x\,dx = C + x\,\coth^{-1}x + \dfrac{1}{2}\log_e(x^2 - 1)$
$$= C + \frac{x+1}{2}\log_e(x + 1) - \frac{x-1}{2}\log_e(x - 1).$$

11. $\int \operatorname{sech}^{-1}x\,dx = C + x\,\operatorname{sech}^{-1}x + \cos^{-1}x.$

12. $\int \operatorname{cosech}^{-1}x\,dx = C + x\,\operatorname{cosech}^{-1}x + \sin^{-1}x.$

VIII. MIXED FUNCTIONS.

1. $\displaystyle\int\frac{\sin x\,dx}{x}=C+x-\frac{x^3}{3.3!}+\frac{x^5}{5.5!}-\frac{x^7}{7.7!}+\cdots.$

2. $\displaystyle\int\frac{\cos x\,dx}{x}=C+\log_e x-\frac{x^2}{2.2!}+\frac{x^4}{4.4!}-\frac{x^6}{6.6!}+\cdots.$

3. $\displaystyle\int\frac{\tan x\,dx}{x}=C+\frac{4(4-1)B_1 x^2}{2.2!}+\frac{4^2(4^2-1)B_2 x^4}{4.4!}+\frac{4^3(4^3-1)B_3 x^6}{6.6!}$
$$+\cdots.^*$$

4. $\displaystyle\int\frac{\cot x\,dx}{x}=C-\frac{1}{x}-\frac{4B_1 x}{1.2!}-\frac{4^2 B_2 x^3}{3.4!}-\frac{4^3 B_3 x^5}{5.6!}-\cdots.^*$

5. $\displaystyle\int\frac{\sec x\,dx}{x}=\frac{1}{2x}\log\frac{1+\sin x}{1-\sin x}+\int\left\{\sin x+\frac{1}{3}\sin^3 x+\frac{1}{5}\sin^5 x+\cdots\right\}\frac{dx}{x^2}.$

Use II. A., II. F., and IX. C. 3 and 4.

6. $\displaystyle\int\frac{\operatorname{cosec} x}{x}dx=C-\frac{1}{x}+2\left\{\frac{(2-1)B_1 x}{1.2!}+\frac{(2^3-1)B_2 x^3}{3.4!}\right.$
$$\left.+\frac{(2^5-1)B_3 x^5}{5.6!}+\cdots\right\}.^*$$

7. $\displaystyle\int\frac{e^x dx}{x}=C+\log_e x+x+\frac{x^2}{2.2!}+\frac{x^3}{3.3!}+\frac{x^4}{4.4!}+\cdots.$

8. $\displaystyle\int x^n e^{ax}dx$ $[n\equiv\text{a positive integer}]$

$$=C+e^{ax}\left\{\frac{x^n}{a}-\frac{nx^{n-1}}{a^2}+\frac{n(n-1)}{a^3}x^{n-2}\right.$$
$$\left.+\cdots\frac{n(n-1)(n-2)\cdots 2.1}{a^{n+1}}\right\}.$$

If n is not a positive integer, use II. D.

9. $\displaystyle\int\frac{e^x}{x^m}dx$ $[m\equiv\text{a positive integer}]$

$$=C-e^x\left\{\frac{1}{(m-1)x^{m-1}}+\frac{1}{(m-1)(m-2)x^{m-2}}+\cdots\right.$$
$$\left.+\frac{1}{(m-1)(m-2)\cdots 2.x}\right\}+\frac{1}{(m-1)!}\int\frac{e^x dx}{x}.\ \ \text{(See VIII. 7.)}$$

Otherwise see IX. C. 7.

---

* For values of $B_1\ B_2\cdots$ see *note* at end of Section VIII.

10. $\int x^{m+nx} dx = \int x^m e^{nx \log_e x} dx$

$$= \int x^m \left\{ 1 + nx \log_e x + \frac{(nx \log_e x)^2}{2!} + \frac{(nx \log_e x)}{3!} + \cdots \right\} dx.$$

Use II. A. and VIII. 18.

11. $\int e^{ax+b} \sin(px+q) dx = C + \dfrac{e^{ax+b}\{a\sin(px+q) - p\cos(px+q)\}}{a^2 + p^2}.$

12. $\int e^{ax+b} \cos(px+q) dx = \int e^{ax+b} \sin(px + \overline{q + \tfrac{\pi}{2}}) dx.$ (See VIII. 11.)

13. $\int x^n \cos^p x \sin^q x\, dx$, where p and q are positive integers.

By II. F. and II. A. reduce to IX. C., 1 and 2.

14. $\int F(x) f(\sin x, \cos x) dx$,   [$F(\ )$ and $f(\ ) \equiv$rational integral functions].

By II. F. and II. A. reduce to IX. C., 1 and 2.

15. $\int \log_e(\sin mx) dx = C - \dfrac{x}{m}\log_e 2 - \dfrac{1}{2m}(\sin 2mx + \dfrac{1}{2^2}\sin 4mx$

$$+ \frac{1}{3^2}\sin 6mx + \frac{1}{4^2}\sin 8mx + \cdots)$$

$$\left[\text{if } o < mx < \frac{\pi}{2} \right].$$

16. $\int \log_e(\cos mx) dx = C - \dfrac{x}{m}\log_e 2 + \dfrac{1}{2m}(\sin 2mx - \dfrac{1}{2^2}\sin 4mx$

$$+ \frac{1}{3^2}\sin 6mx + \frac{1}{4^2}\sin 8mx - \cdots)$$

$$\left[\text{if } -\frac{\pi}{2} < mx < \frac{\pi}{2} \right].$$

17. $\int \log_e(\tan mx) dx = C - \dfrac{1}{m}(\sin 2mx + \dfrac{1}{3^2}\sin 6mx + \dfrac{1}{5^2}\sin 10mx$

$$+ \frac{1}{7^2}\sin 14mx + \cdots).$$

18. $\int x^m (\log_e x)^n dx = \int e^{(m+1)X} X^n dX$

$$X \equiv \log_e x.$$

(See VIII. 8.)

19. $\int x^m F(\log_a x)dx$,   [F( )≡a rational integral function]

$$= \int e^{(m+1)X} F\left(\frac{X}{\log_e a}\right) dX, \ X \equiv \log_e x .$$

Use II. A. and VIII. 8.

Definite Integrals with Particular [or Numerical] Limits.

20. $\int_0^\infty e^{-x} x^n dx = \Gamma(n+1)$.   (See X.)

21. $\int_0^1 x^m (\log x)^n dx = (-1)^n \dfrac{\Gamma(n+1)}{(m+1)^{n+1}}$.   (See X.)

22. $\int_0^1 \dfrac{\log x}{1-x} dx = -\dfrac{\pi^2}{6}$.

23. $\int_0^\infty \dfrac{\sin px}{x} dx = \dfrac{\pi}{2}$ if $p > 0$.

24. $\int_0^\infty \dfrac{\cos px}{x} dx = \infty$.

25. $\int_0^{\pi/2} \log_e(\sin x)dx = -\dfrac{\pi}{2}\log_e 2$.

Note.—Bernoulli's Numbers.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10} etc., etc.
$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$	$\frac{3617}{510}$	$\frac{43867}{798}$	$\frac{122277}{2310}$.

IX.—FORMULAS OF REDUCTION.

Note.—Formulas of Reduction may be obtained by combinations of I. 4, I. 7, and II. A, B, C.

IX. A. *Algebraical.*

$$\int x^m X^r dx \quad \text{where } X \equiv ax^n + b$$

m, n, r being $+$ or $-$, whole or fractional indices.

By the use of one of the subjoined formulæ, the integral of any one of the following 9 functions may be reduced to that of any of the other 8:

$$x^{m+n}X^{r+1} \qquad x^m X^{r+1} \qquad x^{m-n}X^{r+1}$$
$$x^{m+n}X^{r} \qquad x^m X^{r} \qquad x^{m-n}X^{r}$$
$$x^{m+n}X^{r-1} \qquad x^m X^{r-1} \qquad x^{m-n}X^{r-1}.$$

(i) is useful when m and n are of opposite sign; (iii), (iv), (v) are useful when r is $-$;

(iii) and (vi) used together give the reduction from $x^m X^r$ to $x^m X^{r-2}$
(iv) ,, (vi) ,,   ,,   ,, ,, ,, ,, ,, ,, $x^{m-n}X^{r-2}$
(iv) ,, (viii) ,,   ,,   ,, ,, ,, ,, ,, ,, $x^{m-2n}X^{r-2}$.

N.B.—When r is a $+$ integer, this integral can be dealt with by binomial expansion of X^r. In other particular cases the substitutions of III. A. 15 and III. A. 20 may be used.

(i) $\displaystyle \int x^m X^r dx = \frac{1}{(m+1)b} \left\{ x^{m+1}X^{r+1} - (m+1+nr+n)a \int x^{m+n} X^r dx \right\}$

(ii) $\displaystyle = \frac{1}{(m+1+nr)a} \left\{ x^{m+1-n}X^{r+1} - (m+1-n)b \int x^{m-n} X^r dx \right\}$

(iii) $\displaystyle = \frac{1}{(nr+1)b} \left\{ -x^{m+1}X^{r+1} + (m+1+nr+n) \int x^m X^{r+1} dx \right\}$

(iv) $\displaystyle = \frac{x^{m+1}X^{r+2}}{(m+1)b^2} + \frac{a}{n(r+1)b^2} \left\{ x^{m+1+n}X^{r+1} \right.$
$\displaystyle \left. - \frac{(m+1+nr+n)(m+1+nr+2n)}{m+1} \int x^{m+n}X^{r+1} dx \right\}$

(v) $\displaystyle = \frac{1}{n(r+1)a} \left\{ x^{m+1-n}X^{r+1} - (m+1-n) \int x^{m-n}X^{r+1} dx \right\}$

(vi) $\displaystyle = \frac{1}{m+1+nr} \left\{ x^{m+1}X^r + nrb \int x^m X^{r-1} dx \right\}$

(vii) $\int x^m X^r dx = \dfrac{1}{m+1} \left\{ x^{m+1}X^r - nra \int x^{m+n}X^{r-1} dx \right\}$

(viii) $\qquad = \dfrac{1}{(m+1+nr)(m+1+nr-n)} \left\{ (m+1-n)x^{m+1}X^r \right.$

$\qquad\qquad \left. + \dfrac{nr}{a} x^{m+1-n}X^{r+1} - (m+1-n)nr\dfrac{b^2}{a} \int x^{m-n}X^{r-1} dx \right\}$

2. $X \equiv (ax^{2n} + bx^n + c)$

(i) $\int x^m X^r dx = \dfrac{1}{m+1+nr} \left\{ x^{m+1}X^r + nrc \int x^m X^{r-1} dx \right.$

$\qquad\qquad \left. - nra \int x^{m+2n}X^{r-1} dx \right\}$

(ii) $\qquad = \dfrac{1}{(m+1+2nr)a} \left\{ x^{m+1-2n}X^{r+1} \right.$

$\qquad\qquad - (m+1+nr-n)b \int x^{m-n}X^r dx$

$\qquad\qquad \left. - (m+1-2n)c \int x^{m-2n}X^r dx \right.$

3. $X \equiv ax^2 + bx + c$

$\int x^m X^{-\frac{1}{2}} dx = \dfrac{1}{ma} \left\{ x^{m-1}X^{\frac{1}{2}} - (m-\frac{1}{2})b \int x^{m-1}X^{-\frac{1}{2}} dx \right.$

$\qquad\qquad \left. - (m-1)c \int x^{m-2}X^{-\frac{1}{2}} dx \right\}$

If m be a + integer, this reduces to III. B. 14 and 13.

4. $X \equiv ax^2 + bx + c$

$\int X^{-r} dx = \dfrac{1}{(r-1)(4ac - b^2)} \left\{ (2ax + b)X^{-r+1} \right.$

$\qquad\qquad \left. + 2(2n-3)a \int X^{-r+1} dx \right\}$

If r be a + integer, this reduces to III. A. 16, 17.
„ „ (+ integer + $\frac{1}{2}$)   „   „   III. B. 13.

IX. B. *Trigonometrical.*

Note.—The following formulæ remain true when $px + q$ is substituted for x and $p\,dx$ for dx.   (See II. G.)

Sp. Case: If $p = -1$, and $q = \dfrac{\pi}{2}$ radians, in this substitution, we deduce a new formula in which each trigonometrical ratio is replaced by its complementary ratio.

N.B.—The following formulæ, when n and m are integers, reduce to the formulæ referred to in the right-hand column.

1. $\displaystyle \int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1}x \cos x + \frac{n-1}{n}\int \sin^{n-2}x\, dx$ $\left.\begin{array}{l} \text{VI. 1, or} \\ \text{III. A. 1.} \end{array}\right\}$

2. $\displaystyle \int \cos^n x\, dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\, dx$ $\left.\begin{array}{l} \text{VI. 2, or} \\ \text{III. A. 1.} \end{array}\right\}$

3. $\displaystyle \int \frac{dx}{\sin^n x} = \frac{-\cos x}{(n-1)\sin^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2}x}$ $\left.\begin{array}{l} \\ \end{array}\right\}$ VI. 6, or 12.

4. $\displaystyle \int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2}x}$ $\left.\begin{array}{l} \\ \end{array}\right\}$ VI. 5, or 11.

5. $\displaystyle X_{m,\,n} \equiv \int \sin^m x \cos^n x\, dx.$

$\displaystyle X_{m,\,n} = \frac{1}{m+n}\sin^{m+1}x \cos^{n-1}x + \frac{n-1}{m+n}X_{m,\,n-2}$ $\left.\begin{array}{l} \text{VI. 1, 2, or} \\ \text{17, 18, or} \\ \text{III. A. 1.} \end{array}\right\}$

$\displaystyle \qquad\quad = \frac{-1}{m+n}\sin^{m-1}x \cos^{n+1}x + \frac{m-1}{m+n}X_{m-2,\,n}$

6. $\displaystyle X_{m,\,n} \equiv \int \frac{\sin^m x}{\cos^n x}\, dx.$

$\displaystyle X_{m,\,n} = \frac{1}{n-1}\cdot\frac{\sin^{m+1}x}{\cos^{n-1}x} + \frac{n-m-2}{n-1}X_{m,\,n-2}$ $\left.\begin{array}{l} \text{VI. 1, 3, 5,} \\ \text{or 16, or} \\ \text{III. A. 1.} \end{array}\right\}$

$\displaystyle \qquad\quad = -\frac{1}{m-n}\cdot\frac{\sin^{m-1}x}{\cos^{n-1}x} + \frac{m-1}{m-n}X_{m-2,\,n}$

7. $\displaystyle X_{m,\,n} \equiv \int \frac{\cos^n x}{\sin^m x}\, dx$

$\displaystyle X_{m,\,n} = -\frac{1}{m-1}\cdot\frac{\cos^{n+1}x}{\sin^{m-1}x} + \frac{m-n-2}{m-1}X_{m-2,\,n}$ $\left.\begin{array}{l} \text{VI. 1, 2, 4,} \\ \text{or 16, or} \\ \text{III. A. 1.} \end{array}\right\}$

$\displaystyle \qquad\quad = \frac{1}{n-m}\cdot\frac{\cos^{n-1}x}{\sin^{n-1}x} + \frac{n-1}{n-m}X_{m,\,n-2}$

8. $\displaystyle X_{m,\,n} \equiv \int \frac{dx}{\sin^m x \cos^n x}$

$\displaystyle X_{m,\,n} = \frac{1}{n-1}\cdot\frac{1}{\sin^{m-1}x \cos^{n-1}x} + \frac{n+m-2}{n-1}X_{m,\,n-2}$ $\left.\begin{array}{l} \text{VI. 5, 6,} \\ \text{15.} \end{array}\right\}$

$\displaystyle \qquad\quad = -\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1}x \cos^{n-1}x} + \frac{m+n-2}{m-1}X_{m-2,\,n}$

IX. C. *Mixed Functions.*

1. $\displaystyle\int x^n \sin mx\,dx = -\frac{x^n}{m}\cos mx + \frac{n}{m}\int x^{n-1}\cos mx\,dx$ $\left.\begin{array}{l}\text{VI. 1, or 2,}\\ \text{when } n \text{ is a}\\ \text{positive in-}\\ \text{teger. Other-}\\ \text{wise, use II. D.}\\ \text{and II. A.}\end{array}\right\}$

2. $\displaystyle\int x^n \cos mx\,dx = \frac{x^n}{m}\sin mx - \frac{n}{m}\int x^{n-1}\sin mx\,dx$

3. $\displaystyle\int \frac{\sin mx.dx}{x^n} = -\frac{\sin mx}{(n-1)x^{n-1}} + \frac{m}{n-1}\int \frac{\cos mx\,dx}{x^{n-1}}$ $\left.\begin{array}{l}\text{If } n \equiv \text{a posi-}\\ \text{tive integer}\\ \text{VIII. 1 or 2.}\\ \text{Otherwise,}\\ \text{use II. D.}\end{array}\right\}$

4. $\displaystyle\int \frac{\cos mx}{x^n}dx = -\frac{\cos mx}{(n-1)x^{n-1}} - \frac{m}{n-1}\int \frac{\sin mx\,dx}{x^{n-1}}$

5. $\displaystyle\int (\log x)^n x^m dx = \frac{x^{m-1}}{m+1}(\log x)^n - \frac{n}{m+1}\int (\log x)^{n-1}x^m dx$ $\left.\begin{array}{l}\text{III. A. 2 if}\\ n \equiv \text{a posi-}\\ \text{tive integer.}\end{array}\right\}$

6. $\displaystyle\int x^m a^{nx} dx = \frac{x^m a^{nx}}{n\log_e a} - \frac{m}{n\log_e a}\int x^{m-1}a^{nx}dx$ $\left.\begin{array}{l}\text{IV. 2 if } m \equiv\\ \text{a positive}\\ \text{integer.}\end{array}\right\}$

7. $\displaystyle\int \frac{e^x}{x^m}dx = \frac{-e^x}{(m-1)x^{m-1}} + \frac{1}{m-1}\int \frac{e^x dx}{x^{m-1}}$ $\left.\begin{array}{l}\text{VIII. 7 if } m\\ \text{is a whole}\\ \text{number.}\end{array}\right\}$

8. $\displaystyle\int e^{ax}\cos^n x\,dx = \frac{e^{ax}\cos^{n-1}x(n\sin x + a\cos x)}{n^2+a^2}$

$\displaystyle\qquad\qquad\qquad + \frac{n(n-1)}{n^2+a^2}\int e^{ax}\cos^{n-2}x\,dx\,.$ $\left.\begin{array}{l}\text{IV. 2 if } n \equiv\\ \text{even positive}\\ \text{integer.}\\ \text{VIII. 12 if } n\\ \equiv \text{odd positive}\\ \text{integer.}\end{array}\right\}$

9. $\displaystyle\int e^{ax}\sin^n x\,dx = \frac{e^{ax}\sin^{n-1}x(a\sin x - n\cos x)}{n^2+a^2}$

$\displaystyle\qquad\qquad\qquad + \frac{n(n-1)}{n^2+a^2}\int e^{ax}\sin^{n-2}x\,dx$ $\left.\begin{array}{l}\text{IV. 2 if } n \text{ is an even}\\ \text{positive integer.}\\ \text{VIII. 11 if } n \equiv \text{odd}\\ \text{positive integer.}\end{array}\right\}$

10. $\displaystyle\int \frac{e^{ax}}{\cos^n x}dx = -\frac{e^{ax}\{a\cos x - (n-2)\sin x\}}{(n-1)(n-2)\cos^{n-1}x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int \frac{e^{ax}dx}{\cos^{n-2}x}.$

11. $\displaystyle\int \frac{e^{ax}}{\sin^n x}dx = -\frac{e^{ax}\{a\sin x + (n-2)\cos x\}}{(n-1)(n-2)\sin^{n-1}x} + \frac{a^2+(n-2)^2}{(n-1)(n-2)}\int \frac{e^{ax}dx}{\sin^{n-2}x}.$

X.—GAMMA FUNCTIONS.

PROPERTIES OF THE GAMMA FUNCTION, 1 TO 6.

1. Definition: $\Gamma(n) \equiv \int_0^\infty e^{-x} x^{n-1} dx \equiv \int_0^1 \left(\log_e \frac{1}{x}\right)^{n-1} dx$

<div align="center">where $n > o$.</div>

2. $\Gamma(n+1) = n\Gamma(n)$.

3. If n is a positive integer $\Gamma(n) = (n-1)!$ and $\Gamma(1) = 1$.

4. $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\,\pi}$ if $n > o$ and > 1.

5. $\Gamma(n) = \underset{\mu=\infty}{\text{Lim}} \dfrac{\mu!\ \mu^n}{n(n+1)\cdots(n+\mu)}$ where μ is a positive integer.

N.B.—Legendre, in his *Traité des Fonctions Elliptiques*, Vol. II., gives an extended table of 1000 entries from $p = 1$ to $p = 2$, to 12 decimal places.

6. TABLE OF VALUES of $1 + \log_{10}\Gamma p$.

p	0	1	2	3	4	5	6	7	8	9
1·0	·9999	·9753	·9513	·9280	·9053	·8834	·8621	·8415	·8215	·8021
1·1	·7834	·7653	·7478	·7310	·7147	·6990	·6839	·6694	·6554	·6421
1·2	·6292	·6170	·6052	·5940	·5834	·5732	·5636	·5545	·5459	·5878
1·3	·5302	·5231	·5165	·5104	·5047	·4995	·4948	·4905	·4868	·4834
1·4	·4805	·4781	·4761	·4745	·4734	·4726	·4724	·4725	·4731	·4740
1·5	·4755	·4772	·4794	·4820	·4850	·4884	·4921	·4963	·5008	·5057
1·6	·5110	·5167	·5227	·5291	·5359	·5430	·5505	·5583	·5665	·5750
1·7	·5839	·5931	·6027	·6126	·6229	·6335	·6444	·6556	·6672	·6790
1·8	·6913	·7038	·7167	·7299	·7433	·7571	·7712	·7856	·8004	·8154
1·9	·8307	·8463	·8622	·8784	·8949	·9117	·9288	·9462	·9638	·9818

N.B.—To calculate $\Gamma(x)$, when x is given :—

If $x > 1$ and < 2, use the table.

If $x \equiv$ a positive integer, see 3.

If $x > 2$, by using (2) make the value depend on one in which x lies between 1 and 2.

XI. DIFFERENTIAL EQUATIONS.

N.B.—$exp(X) \equiv e^X$.

XI. A. *First Order, First Degree.*

1. $X' + mX = 0$, $X = Ce^{-mx}$, or $x = -\dfrac{1}{m} \log \dfrac{X}{C}$.

2. $X' + f(x) = 0$, $X = C - \displaystyle\int f(x)dx$.

3. $X' + Xf(x) = 0$, $X = C \, exp\{ - \displaystyle\int f(x)dx\}$.

4. $X' + Xf(x) = \phi(x)$

    $X = exp\{ - \displaystyle\int f(x)dx\}[C + \displaystyle\int \phi(x)exp\{\displaystyle\int f(x)dx\}dx]$.

    Sp. Case: $f(x) = k$, $X = e^{-kx}\{C + \displaystyle\int \phi(x)e^{kx}dx\}$.

5. $X' = \phi(x)f(X)$, $\displaystyle\int \dfrac{dX}{f(X)} = \displaystyle\int \phi(x)dx$.

    Sp. Case: $X' = f(X)$, $x = \displaystyle\int \dfrac{dX}{f(X)}$.

6. $f(x,X)X' + \phi(x,X) = 0$. If the condition $\dfrac{\partial \phi}{\partial X} = \dfrac{\partial f}{dx}$ is fulfilled,

    then

    $\displaystyle\int \phi(x,X)dx + \displaystyle\int \left[f(x,X) - \displaystyle\int \dfrac{\partial \phi(x,X)}{\partial X}dx \right] dX = C$

or $\displaystyle\int f(x,X)dX + \displaystyle\int \left[\phi(xX) - \displaystyle\int \dfrac{\partial f(x,X)}{\partial x} \partial x \right] dx = C$

    the integrations being **partial.**

Note.—If the equation as given does not fulfil the above condition, it may do so after being multiplied throughout by a function of x or X or both, called the Integrating Factor. (See Boole, chapters IV., V.). *E.g.* $(x^2X + X + 1) + X'(x + x^3)$.

Integrating Factor, $1/(1 + x^2)$.   Solution: $xX + \tan^{-1}x = C$.

7. $X'f(x,X) + \phi(x,X) = 0$ where $f(x,X) + \phi(x,X)$ is homogeneous in x,X. Substitute $X \equiv x\mathfrak{X}$ and reduce to XI. A. 5.

8. $(ax + bX + c)X' + (fx + gX + h) = 0$.

Assume $ax + bX + c \equiv z$; $fx + gX + h = Z$, hence
$Z'(bZ - az) + fz - gZ = 0$.   (See XI. A. 7.)

Exc. when $a{:}b = f{:}g$, put $\mathfrak{X} \equiv ax + bX$, hence
$(\mathfrak{X} + c)(\mathfrak{X}' - a) + g\mathfrak{X} + bh = 0$.   (See XI. A. 5 Sp. Case.)

9. $X' + Xf(x) = X^n\phi(x)$.   Substitute $\mathfrak{X} \equiv X^{1-n}$.   Then
$\mathfrak{X}' + (1 - n)\mathfrak{X}f(x) = (1 - n)\phi(x)$.   (See XI. A. 4.)

IX. B.   *First Order, Second or Higher Degree.*

1. $f(x, X, X') = 0$.   If possible, solve for X'.   Each solution $X' = \phi(x, X)$, solved by XI. A. if possible, gives part of the general solution.

2. $f(X, X') = 0$.   Use XI. B. 1 if possible: otherwise solve for X if possible.   Each solution $X = \phi(X')$ gives $x = \int \dfrac{\phi'(X')}{X'} dX'$
$+ C$.   Then eliminate X' between the last two equations.

3. $f(x, X') = 0$.   Use XI. B. 1 if possible: otherwise solve for x if possible.   Each solution $x = \phi(X')$ gives $X = \int X'\phi'(X') dX'$
$+ C$.   Then eliminate X' between the last two equations.

4. $X = xX' + f(X')$.   (Clairault's Equation) $X = cx + f(c)$.

XI. C.   *Second Order.*

1. $X'' + m^2X = 0$, $X = A \cos mx + B \sin mx$
$= C \cos (mx + K)$.

2. $X'' - m^2X = 0$, $X = Ae^{mx} + Be^{-mx}$.

3. $X'' + a X' + bX = 0$.   Two forms :—

$$X = e^{-ax/2} \left\{ A exp\left(\frac{x}{2} \sqrt{(a^2 - 4b)} \right) + B exp\left(-\frac{x}{2} \sqrt{(a^2 - 4b)} \right) \right\}$$
$$\text{when } a^2 > 4b$$
$$= e^{-ax/2} \left\{ A \cos\left(\frac{x}{2} \sqrt{4b - a^2} \right) + B \sin\left(\frac{x}{2} \sqrt{4b - a^2} \right) \right\}$$
$$= Ce^{-x/2a} \cos \left\{ \frac{x}{2} \sqrt{(4b - a^2)} + K \right\}$$
$$\text{when } a^2 < 4b.$$

4. $X'' = f(x)$, $X = \int\int f(x) dx dx + Ax + B$.

5. $X'' = f(X)$ $x = \int \dfrac{dX}{\sqrt{\{C + 2\int f(X)dX\}}} + K.$

6. $X'' + aX' + bX = f(x).$
    By XI. C. 3, find $\mathscr{X}$ by solving $\mathscr{X}'' + a\mathscr{X}' + b\mathscr{X} = 0.$
    „ XI. A. 4, „ Ξ „   „ $\mathscr{X}\Xi' + \{2\mathscr{X}' + a\mathscr{X}\}\Xi = f(x).$
    Then $X = \mathscr{X}\int \Xi dx.$

7. $X'' + X'f(x) + XF(x) = \phi(x).$
    Find $\mathscr{X}$ if possible from $\mathscr{X}'' + \mathscr{X}'f(x) + \mathscr{X}F(x) = 0.$
    „ Ξ by XI. A. from $\Xi'\mathscr{X} + \Xi\{2\mathscr{X}' + \mathscr{X}f(x)\} = \phi(x).$
    Then $X = \mathscr{X}\int \Xi dx.$

XI. D. *Order higher than Second.*

1. $f(x, X^{(n-1)}, X^{(n)}) = 0.$ Put $\mathscr{X} = X^{(n-1)}$, and equation becomes $f(x, \mathscr{X}, \mathscr{X}') = 0.$ (See XI. A, or B.)

2. $f(x, X^{(n-2)}, X^{(n-1)}, X^{(n)},) = 0.$ Put $\mathscr{X} = X^{(n-2)}$ and equation becomes $f(x, \mathscr{X}, \mathscr{X}', \mathscr{X}'') = 0.$ (See XI. C.)

3. $X^{(n)} = f(x).$
    $$X = \int^{(n)} f(x)dx^n + C_1 x^{n-1} + C_2 x^{n-2} + \cdots + C_n.$$

4. $X^{(n)} + a_1 X^{(n-1)} + a_2 X^{(n-2)} + \cdots + a_{n-1} X' a_n X = 0$
    $X = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}$
    where $m_1, m_2, \cdots m_n$ are the roots of the auxiliary equation
    $m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_{n-1}m + a_n = 0.$

Note 1.—If $m_1 = p + q\sqrt{-1}$ and $m_2 = p - q\sqrt{-1}$ are a pair of imaginary roots, the terms $C_1 e^{m_1 x} + C_2 e^{m_2 x}$ are equivalent to the *real* form $e^{px}(A \cos qx + B \sin qx).$

Note 2.—If there be r *equal* roots $m_1, m_2 \cdots m_r$, each $= \mu$, the corresponding terms in the value of X are $(C_1 + C_2 x + C_3 x^2 + \cdots + C^r x^{r-1})e^{\mu x}.$ And if there are r pairs of imaginary roots each $= p \pm q\sqrt{-1}$, the terms are

$$e^{px}\{A_1 \cos qx + B_1 \sin qx + x(A_2 \cos qx + B_2 \sin qx) + \cdots$$
$$+ x^{r-1}(A_r \cos qx + B_r \sin qx)\}$$

5. $X^{(n)} + a_1 X^{(n-1)} + a_2 X^{(n-2)} + \cdots + a_n X = b_0 + b_1 x + b_2 x^2 + \cdots + b_r x^r.$

Differentiate both sides $r+1$ times and solve the resulting equation by XI. D. 4. This solution is too general, having $n+r+1$ arbitrary constants: but by substituting in the equation and using I. 11, we get $r+1$ relations between the constants.

Otherwise: see XI. D. 6.

6. $X^{(n)} + a_1 X^{(n-1)} + a_2 X^{(n-2)} + \cdots + a_n X = f(x.)$

Let $X = F(x)$ be the solution on the supposition that $f(x) = 0.$ (See XI. D. 4.)

Then $X = F(x) + \sum\limits_{r=1}^{r=n} A_r exp(m_r x) \int exp\{-m_r x f(x)\}\,dx,$

where $A_1\ A_2 \cdots$ are such as to make the equation

$$\frac{A_1}{m-m_1} + \frac{A_2}{m-m_2} + \cdots + \frac{A_n}{m-m_n} = \frac{1}{m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n}$$

identically true. (See II. E.)

Another method: By variation of Parameters; see Forsyth, § 75.

XI. E. *Partial Differential Equations.*

1. $\dfrac{\partial y}{\partial t} = a^2 \dfrac{\partial^2 y}{\partial x^2},\quad y = C exp(ax + a^2 a^2 t)$

where C and a are arbitrary constants, or $y = \{A \cos ax + B \sin ax\} exp(-a^2 a^2 t)$, A and B arbitrary constants.

General Solution: $y =$ sum of any number of solutions like the above.

E.g. $y = \int_{a_1}^{a_2} F(a) exp(ax + a^2 a^2 t)\,da$, where F is an arbitrary function.

2. $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2},\quad y = F(x + at) + f(x - at)$

where F and f denote arbitrary functions.

3. $a\dfrac{\partial^2 y}{\partial t^2} + b\dfrac{\partial^2 y}{\partial x \partial t} + c\dfrac{\partial^2 y}{\partial x^2} + f\dfrac{\partial y}{\partial t} + g\dfrac{\partial y}{\partial x} + hy - 0y = Ce^{ax+\beta t}$ where C, a, β are arbitrary; but a, β subject to the condition

$$a\beta^2 + ba\beta + ca^2 + f\beta + ga + h^{\cdot} - 0.$$

General Solution: $y =$ the sum of any number of such particular solutions.

4. $a\dfrac{\partial y}{\partial t} + b\dfrac{\partial y}{\partial x} + c = 0$, $(ct + ay) = \phi(cx + by)$

where $\phi \equiv$ an arbitrary function.

A CATALOGUE

OF

SCIENTIFIC AND TECHNICAL WORKS

PUBLISHED BY

CHARLES GRIFFIN & COMPANY,

LIMITED.
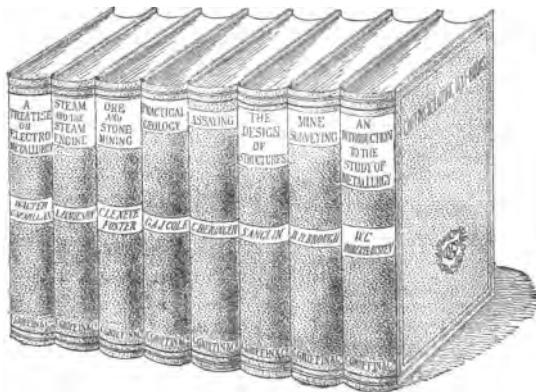
MESSRS. CHARLES GRIFFIN & COMPANY'S
PUBLICATIONS may be obtained through any Bookseller
in the United Kingdom, or will be sent post free on receipt of
remittance to cover published price. To prevent delay, Orders
should be accompanied by a Remittance. Cheques and Postal
Orders to be crossed "SMITH, PAYNE & SMITHS."

☞ GENERAL and MEDICAL CATALOGUES forwarded
Post-free on Application.

LONDON:
EXETER STREET, STRAND.

INDEX TO AUTHORS.

INDEX TO SUBJECTS.

THE DESIGN OF STRUCTURES:

Practical Treatise on the Building of Bridges, Roofs, &c.

By S. ANGLIN, C.E.,

Master of Engineering, Royal University of Ireland, late Whitworth Scholar, &c.

With very numerous Diagrams, Examples, and Tables.
Large Crown 8vo. Cloth.

SECOND EDITION, *Revised.* 16s.

The leading features in Mr. Anglin's carefully-planned "Design of Structures" may be briefly summarised as follows :—

1. It supplies the want, long felt among Students of Engineering and Architecture, of a concise Text-book on Structures, requiring on the part of the reader a knowledge of ELEMENTARY MATHEMATICS only.

2. The subject of GRAPHIC STATICS has only of recent years been generally applied in this country to determine the Stresses on Framed Structures; and in too many cases this is done without a knowledge of the principles upon which the science is founded. In Mr. Anglin's work the system is explained from FIRST PRINCIPLES, and the Student will find in it a valuable aid in determining the stresses on all irregularly-framed structures.

3. A large number of PRACTICAL EXAMPLES, such as occur in the every-day experience of the Engineer, are given and carefully worked out, some being solved both analytically and graphically, as a guide to the Student.

4. The chapters devoted to the practical side of the subject, the Strength of Joints, Punching, Drilling, Rivetting, and other processes connected with the manufacture of Bridges, Roofs, and Structural work generally, are the result of MANY YEARS' EXPERIENCE in the bridge-yard; and the information given on this branch of the subject will be found of great value to the practical bridge-builder.

"Students of Engineering will find this Text-Book INVALUABLE."—*Architect.*

"The author has certainly succeeded in producing a THOROUGHLY PRACTICAL Text-Book."—*Builder.*

"We can unhesitatingly recommend this work not only to the Student, as the BEST TEXT-BOOK on the subject, but also to the professional engineer as an EXCEEDINGLY VALUABLE book of reference."—*Mechanical World.*

"This work can be CONFIDENTLY recommended to engineers. The author has wisely chosen to use as little of the higher mathematics as possible, and has thus made his book of REAL USE TO THE PRACTICAL ENGINEER. . . . After careful perusal, we have nothing but praise for the work."—*Nature.*

CHEMISTRY FOR ENGINEERS AND MANUFACTURERS.

A PRACTICAL TEXT-BOOK.

BY

BERTRAM BLOUNT, AND A. G. BLOXAM,

F.I.C., F.C.S.,
Consulting Chemist to the Crown Agents for the Colonies.

F.I.C., F.C.S.,
Consulting Chemist, Head of the Chemistry Department, Goldsmiths' Inst., New Cross.

With Numerous Illustrations. In Two Volumes, Large 8vo.

Sold Separately.

"The authors have SUCCEEDED beyond all expectation, and have produced a work which should give FRESH POWER to the Engineer and Manufacturer."—*The Times.*

VOLUME I. Price 10s. 6d.

CHEMISTRY OF ENGINEERING, BUILDING, AND METALLURGY.

GENERAL CONTENTS.

I. INTRODUCTION.
   1. Chemistry of the Chief Materials of Construction.
   2. Sources of Energy.
   3. Chemistry of Steam-raising.

4. Chemistry of Lubrication and Lubricants.
II. METALLURGICAL PROCESSES used in the Winning and Manufacture of Metals.

"PRACTICAL THROUGHOUT . . . an ADMIRABLE TEXT-BOOK, useful not only to Students, but to ENGINEERS and MANAGERS OF WORKS in PREVENTING WASTE and IMPROVING PROCESSES."—*Scotsman.*
"EMINENTLY PRACTICAL."—*Glasgow Herald.*
"A book worthy of HIGH RANK . . . its merit is great . . . treatment of the subject of GASEOUS FUEL particularly good. . . . WATER GAS and its production clearly worked out. . . . Altogether a most creditable production. We WARMLY RECOMMEND IT, and look forward with keen interest to the appearance of Vol. II."—*Journal of Gas Lighting.*

VOLUME II. Price 16s.

THE CHEMISTRY OF MANUFACTURING PROCESSES.

GENERAL CONTENTS.

I. Sulphuric Acid Manufacture.
II. Manufacture of Alkali, &c.
III. Destructive Distillation.
IV. Artificial Manure Manufacture.
V. Petroleum.
VI. Lime and Cement.
VII. Clay Industries and Glass.
VIII. Sugar and Starch.
IX. Brewing and Distilling.
X. Oils, Resins, and Varnishes.

XI. Soap and Candles.
XII. Textiles and Bleaching.
XIII. Colouring Matters, Dyeing, and Printing.
XIV. Paper and Pasteboard.
XV. Pigments and Paints.
XVI. Leather, Glue, and Size.
XVII. Explosives and Matches.
XVIII. Minor Chemical Manufactures.

ASSAYING (A Text-Book of):

For the use of Students, Mine Managers, Assayers, &c.

By C. BERINGER, F.C.S.,
Late Chief Assayer to the Rio Tinto Copper Company, London,

And J. J. BERINGER, F.I.C., F.C.S.,
Public Analyst for, and Lecturer to the Mining Association of, Cornwall.

With numerous Tables and Illustrations. Crown 8vo. Cloth, 10/6.
FOURTH EDITION ; Revised.

GENERAL CONTENTS. — PART I. — INTRODUCTORY: MANIPULATION: Sampling; Drying ; Calculation of Results—Laboratory-books and Reports. METHODS : Dry Gravimetric; Wet Gravimetric—Volumetric Assays: Titrometric, Colorimetric. Gasometric— Weighing and Measuring—Reagents—Formulæ, Equations, &c.—Specific Gravity.
PART II.—METALS : Detection and Assay of Silver, Gold, Platinum, Mercury, Copper, Lead, Thallium, Bismuth, Antimony, Iron, Nickel, Cobalt, Zinc, Cadmium, Tin, Tungsten, Titanium, Manganese, Chromium, &c.—Earths, Alkalies.
PART III.—NON-METALS: Oxygen and Oxides; The Halogens—Sulphur and Sulphates—Arsenic, Phosphorus, Nitrogen—Silicon, Carbon, Boron—Useful Tables.

"A REALLY MERITORIOUS WORK, that may be safely depended upon either for systematic instruction or for reference."—*Nature.*
"This work is one of the BEST of its kind. . . . Essentially of a practical character. . . . . Contains all the information that the Assayer will find necessary in the examination of minerals."—*Engineer.*

PHOTOGRAPHY:

ITS HISTORY, PROCESSES, APPARATUS, AND MATERIALS.

Comprising Working Details of all the More Important Methods.

By A. BROTHERS, F.R.A.S.

WITH TWENTY-FOUR FULL PAGE PLATES BY MANY OF THE PROCESSES DESCRIBED, AND ILLUSTRATIONS IN THE TEXT.

In 8vo, Handsome Cloth. Price 18s.

GENERAL CONTENTS. — PART. I. INTRODUCTORY — Historical Sketch; Chemistry and Optics of Photography; Artificial Light.— PART II. Photographic Processes.—PART III. Apparatus.—PART IV. Materials.—PART V.—Applications of Photography ; Practical Hints.

"Mr. Brothers has had an experience in Photography so large and varied that any work by him cannot fail to be interesting and valuable. . . . A MOST COMPREHENSIVE volume, entering with full details into the various processes, and VERY FULLY illustrated. The PRACTICAL HINTS are of GREAT VALUE. . . . Admirably got up."—*Brit. Jour. of Photography.*
"For the Illustrations alone, the book is most interesting ; but, apart from these, the volume is valuable, brightly and pleasantly written, and MOST ADMIRABLY ARRANGED."— *Photographic News.*
"Certainly the FINEST ILLUSTRATED HANDBOOK to Photography which has ever been published. Should be on the reference shelves of every Photographic Society."—*Amateur Photographer.*
"A handbook so far in advance of most others, that the Photographer must not fail to obtain a copy as a reference work."—*Photographic Work.*
"The COMPLETEST HANDBOOK of the art which has yet been published."—*Scotsman.*

WORKS BY A. WYNTER BLYTH, M.R.C.S., F.C.S.,

Barrister-at-Law, Public Analyst for the County of Devon, and Medical Officer of Health for St. Marylebone.

FOODS:

THEIR COMPOSITION AND ANALYSIS.

In Demy 8vo, with Elaborate Tables, Diagrams, and Plates. Handsome Cloth. FOURTH EDITION. Price 21s.

GENERAL CONTENTS.

History of Adulteration—Legislation, Past and Present—Apparatus useful to the Food-Analyst—"Ash"—Sugar—Confectionery—Honey—Treacle—Jams and Preserved Fruits—Starches—Wheaten-Flour—Bread—Oats—Barley—Rye—Rice—Maize—Millet—Potato—Peas—Chinese Peas—Lentils—Beans—MILK—Cream—Butter—Oleo-Margarine—Butterine—Cheese—Lard—Tea—Coffee—Cocoa and Chocolate—Alcohol—Brandy—Rum—Whisky—Gin—Arrack—Liqueurs—Absinthe—Principles of Fermentation—Yeast—Beer—Wine—Vinegar—Lemon and Lime Juice—Mustard—Pepper—Sweet and Bitter Almond—Annatto—Olive Oil—WATER—Standard Solutions and Reagents. *Appendix:* Text of English and American Adulteration Acts.

PRESS NOTICES OF THE FOURTH EDITION.

"Simply INDISPENSABLE in the Analyst's laboratory."—*The Lancet.*
"THE STANDARD WORK on the subject. . . . Every chapter and every page gives abundant proof of the strict revision to which the work has been subjected. . . . The section on MILK is, we believe, the most exhaustive study of the subject extant. . . . An INDISPENSABLE MANUAL for Analysts and Medical Officers of Health."—*Public Health.*
"A new edition of Mr. Wynter Blyth's Standard work, ENRICHED WITH ALL THE RECENT DISCOVERIES AND IMPROVEMENTS, will be accepted as a boon."—*Chemical News.*

POISONS:

THEIR EFFECTS AND DETECTION.

THIRD EDITION. In Large 8vo, Cloth, with Tables and Illustrations. Price 21s.

GENERAL CONTENTS.

I.—Historical Introduction. II.—Classification—Statistics—Connection between Toxic Action and Chemical Composition—Life Tests—General Method of Procedure—The Spectroscope—Examination of Blood and Blood Stains. III.—Poisonous Gases. IV.—Acids and Alkalies. V.—More or less Volatile Poisonous Substances. VI.—Alkaloids and Poisonous Vegetable Principles. VII.—Poisons derived from Living or Dead Animal Substances. VIII.—The Oxalic Acid Group. IX.—Inorganic Poisons. *Appendix:* Treatment, by Antidotes or otherwise, of Cases of Poisoning.

"Undoubtedly THE MOST COMPLETE WORK on Toxicology in our language."—*The Analyst (on the Third Edition).*
"As a PRACTICAL GUIDE, we know NO BETTER work."—*The Lancet (on the Third Edition).*
* In the THIRD EDITION, Enlarged and partly Re-written, NEW ANALYTICAL METHODS have been introduced, and the CADAVERIC ALKALOIDS, or PTOMAINES, bodies playing so great a part in Food-poisoning and in the Manifestations of Disease, have received special attention.

MINE-SURVEYING (A Text-Book of):

For the use of Managers of Mines and Collieries, Students at the Royal School of Mines, &c.

By BENNETT H. BROUGH, F.G.S.,
Late Instructor of Mine-Surveying, Royal School of Mines.

With Diagrams. FIFTH EDITION. Crown 8vo. Cloth, 7s. 6d.

GENERAL CONTENTS.

General Explanations—Measurement of Distances—Miner's Dial—Variation of the Magnetic-Needle—Surveying with the Magnetic-Needle in presence of Iron—Surveying with the Fixed Needle—German Dial—Theodolite—Traversing Underground—Surface-Surveys with Theodolite—Plotting the Survey—Calculation of Areas—Levelling—Connection of Underground- and Surface-Surveys—Measuring Distances by Telescope—Setting-out—Mine-Surveying Problems—Mine Plans—Applications of Magnetic-Needle in Mining—*Appendices.*

" It is the kind of book which has long been wanted, and no English-speaking Mine Agent or Mining Student will consider his technical library complete without it."—*Nature.*

" SUPPLIES A LONG-FELT WANT."—*Iron.*

" A valuable accessory to Surveyors in every department of commercial enterprise."—*Colliery Guardian.*

WORKS
By WALTER R. BROWNE, M.A., M. INST. C.E.,
Late Fellow of Trinity College, Cambridge.

THE STUDENT'S MECHANICS:
An Introduction to the Study of Force and Motion.
With Diagrams. Crown 8vo. Cloth, 4s. 6d.

" Clear in style and practical in method, 'THE STUDENT'S MECHANICS' is cordially to be commended from all points of view."—*Athenæum.*

FOUNDATIONS OF MECHANICS.
Papers reprinted from the *Engineer.* In Crown 8vo, 1s.

FUEL AND WATER:
A Manual for Users of Steam and Water.
By PROF. SCHWACKHÖFER AND W. R. BROWNE, M.A. (See p. 41.)

GAS MANUFACTURE

(THE CHEMISTRY OF).

A Hand-Book on the Production, Purification, and Testing of Illuminating Gas, and the Assay of the Bye-Products of Gas Manufacture. For the Use of Students.

BY

W. J. ATKINSON BUTTERFIELD, M.A., F.C.S.,

Head Chemist, Gas Works, Beckton, London. E.

With Numerous Illustrations. Handsome Cloth. Price 9s.

"The BEST WORK of its kind which we have ever had the pleasure of reviewing."—*Journal of Gas Lighting.*

GENERAL CONTENTS.

⁎ This work deals primarily with the ordinary processes of GAS MANUFACTURE employed in this country, and aims especially at indicating the *principles* on which they are based. The more modern, but as yet subsidiary, processes are fully treated also. The Chapters on *Gas Analysis* and *Photometry* will enable the consumer to grasp the methods by which the quality of the gas he uses is ascertained, and in the Chapter on *The Applications of Gas*, not only is it discussed as an illuminant, but also as a ready source of heat and power. In the final Chapter, an attempt has been made to trace in a readily-intelligible manner the extraction of the principal derivatives from the crude BYE-PRODUCTS. The work deals incidentally with the most modern developments of the industry, including *inter alia* the commercial production and uses of acetylene and the application of compressed gas for Street Traction. The needs of the Students in Technical Colleges and Classes have throughout been kept in view.

LONDON: EXETER STREET, STRAND.

Works by GRENVILLE A. J. COLE, M.R.I.A., F.G.S.,

Professor of Geology in the Royal College of Science for Ireland.

PRACTICAL GEOLOGY

(AIDS IN):

WITH A SECTION ON PALÆONTOLOGY.

SECOND EDITION, Revised.   With Illustrations.   Cloth, 10s. 6d.

GENERAL CONTENTS.—PART I.—SAMPLING OF THE EARTH'S CRUST.   PART II.—EXAMINATION OF MINERALS.   PART III.—EXAMINA-TION OF ROCKS.   PART IV.—EXAMINATION OF FOSSILS.

"Prof. Cole treats of the examination of minerals and rocks in a way that has never been attempted before . . . DESERVING OF THE HIGHEST PRAISE.   Here indeed are 'Aids' INNUMERABLE and INVALUABLE.   All the directions are given with the utmost clear-ness and precision."—*Athenæum.*

"To the younger workers in Geology, Prof. Cole's book will be as INDISPENSABLE as a dictionary to the learners of a language."—*Saturday Review.*

"That the work deserves its title, that it is full of 'AIDS,' and in the highest degree 'PRACTICAL,' will be the verdict of all who use it."—*Nature.*

"This EXCELLENT MANUAL . . . will be A VERY GREAT HELP. . . .   The section on the Examination of Fossils is probably the BEST of its kind yet published. . . .   FULL of well-digested information from the newest sources and from personal research."—*Annals of Nat. History.*

New and Attractive Work by Prof. COLE.

OPEN-AIR STUDIES:

An Introduction to Geology Out-of-doors.

With 12 Full-Page Illustrations from Photographs.   Cloth.   8s. 6d.

GENERAL CONTENTS.—The Materials of the Earth—A Mountain Hollow —Down the Valley—Along the Shore—Across the Plains—Dead Volcanoes —A Granite Highland—The Annals of the Earth—The Surrey Hills—The Folds of the Mountains.

"The FASCINATING 'OPEN-AIR STUDIES' of PROF. COLE give the subject a GLOW OF ANIMATION . . .   cannot fail to arouse keen interest in geology."—*Geological Magazine.*

"EMINENTLY READABLE . . .   every small detail in a scene touched with a sym-pathetic kindly pen that reminds one of the lingering brush of a Constable."—*Nature.*

"The work of Prof. Cole combines ELEGANCE of STYLE with SCIENTIFIC THOROUGHNESS."— *Petermann's Mittheilungen.*

"The book is worthy of its title: from cover to cover it is STRONG with bracing freshness of the mountain and the field, while its ACCURACY and THOROUGHNESS show that it is the work of an earnest and conscientious student. . . .   Full of picturesque touches which are most welcome "—*Natural Science.*

"A CHARMING BOOK, beautifully illustrated."—*Athenæum.*

SEWAGE DISPOSAL WORKS:

A Guide to the Construction of Works for the Prevention of the Pollution by Sewage of Rivers and Estuaries.

BY

W. SANTO CRIMP, M.INST.C.E., F.G.S.,

Late Assistant-Engineer, London County Council.

With Tables, Illustrations in the Text, and 37 Lithographic Plates. **Medium** 8vo. Handsome Cloth.

SECOND EDITION, REVISED AND ENLARGED. 30s.

PART I.—INTRODUCTORY.

Introduction.
Details of River Pollutions and Recommendations of Various Commissions.
Hourly and Daily Flow of Sewage.
The Pail System as Affecting Sewage.
The Separation of Rain-water from the Sewage Proper.

Settling Tanks.
Chemical Processes.
The Disposal of Sewage-sludge.
The Preparation of Land for Sewage Disposal.
Table of Sewage Farm Management.

PART II.—SEWAGE DISPOSAL WORKS IN OPERATION—THEIR CONSTRUCTION, MAINTENANCE, AND COST.

Illustrated by Plates showing the General Plan and Arrangement adopted in each District.

Map of the LONDON Sewage System.
Crossness Outfall.
Barking Outfall.
Doncaster Irrigation Farm.
Beddington Irrigation Farm, Borough of Croydon.
Bedford Sewage Farm Irrigation.
Dewsbury and Hitchin Intermittent Filtration.
Merton, Croydon Rural Sanitary Authority.
Swanwick, Derbyshire.
The Ealing Sewage Works.
Chiswick.
Kingston-on-Thames, A. B. C. Process.
Salford Sewage Works.

Bradford, Precipitation.
New Malden, Chemical Treatment and Small Filters.
Friern Barnet.
Acton, Ferozone and Polarite Process.
Ilford, Chadwell, and Dagenham Works.
Coventry.
Wimbledon.
Birmingham.
Margate.
Portsmouth.
BERLIN Sewage Farms.
Sewage Precipitation Works, Dortmund (Germany).
Treatment of Sewage by Electrolysis.

. From the fact of the Author's having, for some years, had charge of the Main Drainage Works of the Northern Section of the Metropolis, the chapter on LONDON will be found to contain many important details which would not otherwise have been available.

"All persons interested in Sanitary Science owe a debt of gratitude to Mr. Crimp. . . . His work will be especially useful to SANITARY AUTHORITIES and their advisers . . . EMINENTLY PRACTICAL AND USEFUL . . . gives plans and descriptions of MANY OF THE MOST IMPORTANT SEWAGE WORKS of England . . . with very valuable information as to the COST of construction and working of each. . . . The carefully-prepared drawings permit of an easy comparison between the different systems."—*Lancet.*

"Probably the MOST COMPLETE AND BEST TREATISE on the subject which has appeared in our language . . . Will prove of the greatest use to all who have the problem of Sewage Disposal to face."—*Edinburgh Medical Journal.*

LONDON: EXETER STREET, STRAND.

Griffin's "Pocket-Book" Series.

Pocket Size.   Leather.   With Illustrations.   12s. 6d.

HYGIENE (A Hand-Book of),

BY

SURGEON-MAJOR A. M. DAVIES, D.P.H.Camb.,

Late Assistant-Professor of Hygiene, Army Medical School.

General Contents.

Air and Ventilation—Water and Water Supply—Food and Dieting—
Removal and Disposal of Sewage—Habitations—Personal Hygiene—Soils
and Sites—Climate and Meteorology—Causation and Prevention of Disease
—Disinfection.

"This ADMIRABLE HANDBOOK . . . gives FULL information compressed into the smallest
possible bulk."—*Edin. Med. Journal.*
"The elegant dress of the little volume before us is but the outer covering of A TRULY RICH
KERNEL and justly merits the praise it spontaneously calls forth. Attractive to the eye, Surgeon-
Major DAVIES' volume is equally attractive to the mind. Students will find that its 590 pages
comprise ALL information necessary. COMPACT, HANDY, COMPREHENSIVE, it certainly merits a
high place among the text-books of the day."—*Sanitary Record.*
"We are glad to welcome Surgeon-Major Davies' book . . . he has had ample opportunity
to make himself A MASTER OF THE SCIENCE, and he has a right to speak. . . . WONDERFULLY
WELL UP TO DATE, well and clearly written, pleasant to read."—*The Lancet.*
"Really an ADMIRABLE BOOK. . . . A MOST HANDY WORK OF REFERENCE full of information."
—*The Hospital.*
"A singularly compact and elegant volume . . . contains an admirable *précis* of everything
relating to Hygiene CLEARLY and LOGICALLY ARRANGED and easy of reference. Likely, we think,
to be the favourite text-book."—*Public Health.*

Large 8vo.   Handsome Cloth.   12s. 6d.

BLEACHING & CALICO-PRINTING.

A Short Manual for Students and Practical Men.

By GEORGE DUERR,

Director of the Bleaching, Dyeing, and Printing Department at the Accrington and Bacup
Technical Schools; Chemist and Colourist at the Irwell Print Works.

ASSISTED BY WILLIAM TURNBULL

(of Turnbull & Stockdale, Limited).

With Illustrations and upwards of One Hundred Dyed and Printed Patterns
designed specially to show various Stages of the Processes described.

"When a READY WAY out of a difficulty is wanted, it is IN BOOKS LIKE THIS that it is found."—
Textile Recorder.
"Mr. DUERR'S WORK will be found MOST USEFUL . . . The information given of GREAT
VALUE. . . . The Recipes THOROUGHLY PRACTICAL."—*Textile Manufacturer.*

WORKS

By J. R. AINSWORTH DAVIS, B.A.,

PROFESSOR OF BIOLOGY, UNIVERSITY COLLEGE, ABERYSTWYTH;
EXAMINER IN ZOOLOGY, UNIVERSITY OF ABERDEEN.

DAVIS (Prof. Ainsworth): BIOLOGY (An Elementary Text-Book of). In large Crown 8vo, Cloth. SECOND EDITION.

PART I. VEGETABLE MORPHOLOGY AND PHYSIOLOGY. With Complete Index-Glossary and 128 Illustrations. Price 8s. 6d.
PART II. ANIMAL MORPHOLOGY AND PHYSIOLOGY. With Complete Index-Glossary and 108 Illustrations. Price 10s. 6d.

EACH PART SOLD SEPARATELY.

*** NOTE—The SECOND EDITION has been thoroughly Revised and Enlarged, and includes all the leading selected TYPES in the various Organic Groups.

"Certainly THE BEST 'BIOLOGY' with which we are acquainted. It owes its pre-eminence to the fact that it is an EXCELLENT attempt to present Biology to the Student as a CORRELATED AND COMPLETE SCIENCE. The glossarial Index is a MOST USEFUL addition."—*British Medical Journal.*
" Furnishes a CLEAR and COMPREHENSIVE exposition of the subject in a SYSTEMATIC form."—*Saturday Review.*
" Literally PACKED with information."—*Glasgow Medical Journal*

DAVIS (Prof. Ainsworth): THE FLOWERING PLANT, as Illustrating the First Principles of Botany. Large Crown 8vo, with numerous Illustrations. 3s. 6d. SECOND EDITION.

" It would be hard to find a Text-book which would better guide the student to an accurate knowledge of modern discoveries in Botany. . . . The SCIENTIFIC ACCURACY of statement, and the concise exposition of FIRST PRINCIPLES make it valuable for educational purposes. In the chapter on the Physiology of Flowers, an *admirable résumé* is given, drawn from Darwin, Hermann Müller, Kerner, and Lubbock, of what is known of the Fertilization of Flowers."

DAVIS and SELENKA: A ZOOLOGICAL POCKET-BOOK; Or, Synopsis of Animal Classification. Comprising Definitions of the Phyla, Classes, and Orders, with Explanatory Remarks and Tables. By Dr. Emil Selenka, Professor in the University of Erlangen. Authorised English translation from the Third German Edition. In Small Post 8vo, Interleaved for the use of Students. Limp Covers, 4s.

" Dr. Selenka's Manual will be found useful by all Students of Zoology. It is a COMPRE-HENSIVE and SUCCESSFUL attempt to present us with a scheme of the natural arrangement of the animal world."—*Edin. Med. Journal.*
" Will prove very serviceable to those who are attending Biology Lectures. . . . The translation is accurate and clear."—*Lancet.*

LONDON: EXETER STREET, STRAND.

GAS, OIL, AND AIR ENGINES:

A Practical Text - Book on Internal Combustion Motors without Boiler.

BY BRYAN DONKIN, M. INST. C. E.

SECOND EDITION, Revised throughout and Enlarged. With numerous additional Illustrations. Large 8vo. 25s.

GENERAL CONTENTS.—Gas Engines:—General Description—History and Development—British, French, and German Gas Engines—Gas Production for Motive Power—Theory of the Gas Engine—Chemical Composition of Gas in Gas Engines—Utilisation of Heat—Explosion and Combustion. Oil Motors:—History and Development—Various Types—Priestman's and other Oil Engines. Hot-Air Engines:—History and Development—Various Types: Stirling's, Ericsson's, &c., &c.

"The BEST BOOK NOW PUBLISHED on Gas, Oil, and Air Engines. . . . Will be of VERY GREAT INTEREST to the numerous practical engineers who have to make themselves familiar with the motor of the day. . . . Mr. Donkin has the advantage of LONG PRACTICAL EXPERIENCE, combined with HIGH SCIENTIFIC AND EXPERIMENTAL KNOWLEDGE, and an accurate perception of the requirements of Engineers."—*The Engineer.*

"The intelligence that Mr. BRYAN DONKIN has published a Text-book should be GOOD NEWS to all who desire reliable, up-to-date information. . . . His book is MOST TIMELY, and we welcomed it at first sight as being just the kind of book for which everybody interested in the subject has been looking. . . . We HEARTILY RECOMMEND Mr. Donkin's work. . . . A monument of careful labour. . . . Luminous and comprehensive. . . . Nothing of any importance seems to have been omitted."—*Journal of Gas Lighting.*

INORGANIC CHEMISTRY (A Short Manual of).

BY A. DUPRÉ, Ph. D., F. R. S., AND WILSON HAKE,
Ph. D., F. I. C., F. C. S., of the Westminster Hospital Medical School.

SECOND EDITION, Revised.   Crown 8vo.   Cloth, 7s. 6d.

"A well-written, clear and accurate Elementary Manual of Inorganic Chemistry. . . . We agree heartily in the system adopted by Drs. Dupré and Hake. WILL MAKE EXPERIMENTAL WORK TREBLY INTERESTING BECAUSE INTELLIGIBLE."—*Saturday Review.*
"There is no question that, given the PERFECT GROUNDING of the Student in his Science, the remainder comes afterwards to him in a manner much more simple and easily acquired. The work IS AN EXAMPLE OF THE ADVANTAGES OF THE SYSTEMATIC TREATMENT of a Science over the fragmentary style so generally followed. BY A LONG WAY THE BEST of the small Manuals for Students."—*Analyst.*

HINTS ON THE PRESERVATION OF FISH,

IN REFERENCE TO FOOD SUPPLY.

BY J. COSSAR EWART, M. D., F. R. S. E.,
Regius Professor of Natural History, University of Edinburgh.

In Crown 8vo. Wrapper, 6d.

SECOND EDITION, *Revised.* *Royal 8vo.* *With numerous Illustrations and*
13 *Lithographic Plates.* *Handsome Cloth.* *Price* 30s.

BRIDGE-CONSTRUCTION

(A PRACTICAL TREATISE ON):

Being a Text-Book on the Construction of Bridges in Iron and Steel.

FOR THE USE OF STUDENTS, DRAUGHTSMEN, AND ENGINEERS.

By T. CLAXTON FIDLER, M. INST. C.E.,

Prof. of Engineering, University College, Dundee.

"Mr. FIDLER'S SUCCESS arises from the combination of EXPERIENCE and SIMPLICITY OF TREATMENT displayed on every page. . . . Theory is kept in subordination to practice, and his book is, therefore, as useful to girder-makers as to students of Bridge Construction."—("*The Architect*" *on the Second Edition.*)

"Of late years the American treatises on Practical and Applied Mechanics have taken the lead . . . since the opening up of a vast continent has given the American engineer a number of new bridge-problems to solve . . . but we look to the PRESENT TREATISE ON BRIDGE-CONSTRUCTION, and the Forth Bridge, to bring us to the front again."—*Engineer.*

"One of the VERY BEST RECENT WORKS on the Strength of Materials and its application to Bridge-Construction. . . Well repays a careful Study."—*Engineering.*

"An INDISPENSABLE HANDBOOK for the practical Engineer."—*Nature.*

HOW PLANTS LIVE AND WORK:

A Simple Introduction to Real Life in the Plant-world, Based on Lessons originally given to Country Children.

·By ELEANOR HUGHES-GIBB.

With Illustrations. Crown 8vo. Cloth. 2s. 6d.

. The attention of all interested in the Scientific Training of the Young is requested to this DELIGHTFULLY FRESH and CHARMING LITTLE BOOK. It ought to be in the hands of every Mother and Teacher throughout the land.

"The child's attention is first secured, and then, in language SIMPLE, YET SCIENTIFICALLY ACCURATE, the first lessons in plant-life are set before it."—*Natural Science.*

"In every way well calculated to make the study of Botany ATTRACTIVE to the young."—*Scotsman.*

LONDON: **EXETER STREET, STRAND.**

ORE & STONE MINING.

BY

C. LE NEVE FOSTER, D.Sc., F.R.S.,

PROFESSOR OF MINING, ROYAL COLLEGE OF SCIENCE; H.M. INSPECTOR OF MINES.

SECOND EDITION.   With Frontispiece and 716 Illustrations.   34s.

"Dr. Foster's book was expected to be EPOCH-MAKING, and it fully justifies such expectation. . . . A MOST ADMIRABLE account of the mode of occurrence of practically ALL KNOWN MINERALS. Probably stands UNRIVALLED for completeness."—*The Mining Journal.*

GENERAL CONTENTS.

INTRODUCTION. Mode of Occurrence of Minerals: Classification: Tabular Deposits, Masses—Examples: Alum, Amber, Antimony, Arsenic, Asbestos, Asphalt, Barytes, Borax, Boric Acid, Carbonic Acid, Clay, Cobalt Ore, Copper Ore, Diamonds, Flint, Freestone, Gold Ore, Graphite, Gypsum, Ice, Iron Ore, Lead Ore, Manganese Ore, Mica, Natural Gas, Nitrate of Soda, Ozokerite, Petroleum, Phosphate of Lime Potassium Salts, Quicksilver Ore, Salt, Silver Ore, Slate, Sulphur, Tin Ore, Zinc Ore. Faults. **Prospecting:** Chance Discoveries — Adventitious Finds — Geology as a Guide to Minerals—Associated Minerals—Surface Indications. **Boring:** Uses of Bore-holes—Methods of Boring Holes: i. By Rotation, ii. By Percussion with Rods, iii. By Percussion with Rope. **Breaking Ground:** Hand Tools—Machinery— Transmission of Power—Excavating Machinery: i. Steam Diggers, ii. Dredges, iii. Rock Drills, iv. Machines for Cutting Grooves, v. Machines for Tunnelling— Modes of using Holes—Driving and Sinking—Fire-setting—Excavating by Water. **Supporting Excavations:** Timbering—Masonry—Metallic Supports—Watertight Linings—Special Processes. **Exploitation:** Open Works:—Hydraulic Mining— Excavation of Minerals under Water—Extraction of Minerals by Wells and Bore-holes—Underground Workings—Beds—Veins—Masses. **Haulage or Transport:** Underground: by Shoots, Pipes, Persons, Sledges, Vehicles, Railways, Machinery, Boats—Conveyance above Ground. **Hoisting or Winding:** Motors, Drums, and Pulley Frames—Ropes, Chains, and Attachments—Receptacles—Other Appliances— Safety Appliances—Testing Ropes—Pneumatic Hoisting. **Drainage:** Surface Water —Dams—Drainage Tunnels—Siphons— Winding Machinery — Pumping Engines above ground—Pumping Engines below ground—Co-operative Pumping. **Ventilation:** Atmosphere of Mines—Causes of Pollution of Air—Natural Ventilation— Artificial Ventilation: i. Furnace Ventilation, ii. Mechanical Ventilation—Testing the Quality of Air—Measuring the Quantity and Pressure of the Air—Efficiency of Ventilating Appliances — Resistance caused by Friction. **Lighting:** Reflected Daylight—Candles—Torches—Lamps—Wells Light—Safety Lamps—Gas—Electric Light. **Descent and Ascent:** Steps and Slides—Ladders—Buckets and Cages—Man Engine. **Dressing:** i. Mechanical Processes: Washing, Hand Picking, Breaking Up, Consolidation, Screening—ii. Processes depending on Physical Properties: Motion in Water, Motion in Air—Desiccation—Liquefaction and • Distillation— Magnetic Attraction—iii. Chemical Processes: Solution, Evaporation and Crystallisation, Atmospheric Weathering, Calcination, Cementation, Amalgamation—Application of Processes—Loss in Dressing—Sampling. **Principles of Employment of Mining Labour:** Payment by Time, Measure, or Weight—By Combination of these — By Value of Product. **Legislation affecting Mines and Quarries:** Ownership—Taxation—Working Regulations—Metalliferous Mines Regulation Acts —Coal Mines Regulation Act—Other Statutes. **Condition of the Miner:** Clothing —Housing—Education—Sickness—Thrift—Recreation. **Accidents:** Death Rate of Miners from Accidents—Relative Accident Mortality Underground and Above-ground—Classification of Accidents—Ambulance Training.

"This EPOCH-MAKING work . . . appeals to MEN OF EXPERIENCE no less than to students . . . gives numerous examples from the MINING PRACTICE of EVERY COUNTRY. Many of its chapters are upon subjects not usually dealt with in text books. . . . Of great interest. . . . Admirably illustrated."—*Berg- und Hüttenmännische Zeitung.*
"This SPLENDID WORK."—*Oesterr. Ztschrft. für Berg- und Hüttenwesen.*

LONDON: EXETER STREET, STRAND.

EDITION for 1896.

Cloth, for Office use, 8s. 6d.   Leather, for the Pocket, 8s. 6d.

⌈GRIFFIN'S ELECTRICAL PRICE-BOOK.

For Electrical, Civil, Marine, and Borough Engineers, Local Authorities, Architects, Railway Contractors, &c., &c.

.EDITED BY H. J. DOWSING,

Member of the Institution of Electrical Engineers; of the Society of Arts; of the London Chamber of Commerce, &c.

PART I.—PRICES AND DETAILS OF MACHINERY AND APPARATUS.
PART II.—USEFUL INFORMATION CONCERNING THE SUPPLY OF ELECTRICAL ENERGY; Complete Estimates; Reports, Rules and Regulations, Useful Tables, &c.; and General Information regarding the carrying out of Electrical Work.

"The ELECTRICAL PRICE-BOOK REMOVES ALL MYSTERY about the cost of Electrical Power. By its aid the EXPENSE that will be entailed by utilising electricity on a large or small scale can be discovered. . . . Contains that sort of information which is most often required in an architect's office when the application of Electricity is being considered."—*Architect.*

"The value of this Electrical Price-Book CANNOT BE OVER-ESTIMATED. . . . Will save time and trouble both to the engineer and the business man."—*Machinery.*

*** The Publishers beg to call attention to the New Edition of the ELECTRICAL ENGINEERS' PRICE-BOOK. It is gratifying to learn that the work has been found USEFUL and VALUABLE by the general body of Engineers, Architects, Builders, and others. The prices given for work carried out have been acknowledged FAIR, and such as provide for RELIABLE ;AND GOOD WORKMANSHIP, and they have proved of use in Arbitrations, Appeals, and Assessment Cases.

GRIFFIN (John Joseph, F.C.S.):

CHEMICAL RECREATIONS: A Popular Manual of Experimental Chemistry. With 540 Engravings of Apparatus. *Tenth Edition.* Crown 8vo. Cloth. Complete in one volume, cloth, gilt top, 12/6.

Part I.—Elementary Chemistry, 2/.
Part II.—The Chemistry of the Non-Metallic Elements, 10/6.

GURDEN (Richard Lloyd, Authorised Surveyor

for the Governments of New South Wales and Victoria):

TRAVERSE TABLES: computed to Four Places Decimals for every Minute of Angle up to 100 of Distance. For the use of Surveyors and Engineers. FOURTH EDITION. Folio, strongly half-bound, 21/.

*** *Published with Concurrence of the Surveyors-General for New South Wales and Victoria.*

"Those who have experience in exact SURVEY-WORK will best know how to appreciate the enormous amount of labour represented by this valuable book. The computations enable the user to ascertain the sines and cosines for a distance of twelve miles to within half an inch, and this BY REFERENCE TO BUT ONE TABLE, in place of the usual Fifteen minute computations required. This alone is evidence of the assistance which the Tables ensure to every user, and as every Surveyor in active practice has felt the want of such assistance, few knowing of their publication will remain without them."—*Engineer.*

Griffin's Standard Publications

FOR

ENGINEERS, ELECTRICIANS, ARCHITECTS, BUILDERS, NAVAL CONSTRUCTORS, AND SURVEYORS.

GRIFFIN'S NAUTICAL SERIES.

EDITED BY EDW. BLACKMORE,

Master Mariner, First Class Trinity House Certificate, Assoc. Inst. N.A.;
AND WRITTEN, MAINLY, by SAILORS for SAILORS.

In Crown 8vo.   With Illustrations and Plates.

"A VERY USEFUL SERIES."—*Nature.*   "This ADMIRABLE SERIES."—*Fairplay.*
"The volumes of MESSRS. GRIFFIN'S NAUTICAL SERIES may well and profitably be
read by ALL interested in our NATIONAL MARITIME PROGRESS."—*Marine Engineer.*
"EVERY SHIP should have the WHOLE SERIES as a REFERENCE LIBRARY. HAND-
SOMELY and STRONGLY BOUND, CLEARLY PRINTED and ILLUSTRATED."—*Liverpool Journ.
of Commerce.*

Know Your Own Ship: A Simple Explanation of the Stability, Con-
struction, Tonnage, and Freeboard of Ships. By THOS. WALTON, Naval Architect,
Lecturer to Ships' Officers, Government Navigation School, Leith. With numerous
Illustrations. SECOND EDITION. 5s.

"MR. WALTON'S book will be found VERY USEFUL."—*The Engineer.*
"Will attain LASTING SUCCESS . . . EXCEEDINGLY HANDY."—*Shipping World.*

Latitude and Longitude: How to find them. By W. J. MILLAR,
C.E., late Sec. to the Inst. of Engineers and Shipbuilders in Scotland. 2s.

"Cannot but prove an acquisition to those studying Navigation."—*Marine Engineer.*
"Young Seamen will find it HANDY and USEFUL, SIMPLE and CLEAR."—*The Engineer.*

Practical Mechanics: Applied to the requirements of the Sailor.
By THOS. MACKENZIE, Master Mariner, F.R.A.S. 3s. 6d.

"THIS EXCELLENT BOOK . . . contains a LARGE AMOUNT of information."—*Nature.*
"WELL WORTH the money . . . EXCEEDINGLY HELPFUL."—*Shipping World.*
"NO SHIPS' OFFICERS' BOOKCASE will henceforth be complete without CAPTAIN
MACKENZIE'S 'PRACTICAL MECHANICS.' Notwithstanding my many years' experience
at sea, it has told me *how much more there is to acquire.*"—(Letter to the Publishers
from a Master Mariner).

The British Mercantile Marine; An Historical Sketch of its Rise
and Development. By the EDITOR. 3s. 6d.

Elementary Seamanship.   By D. WILSON-BARKER, Master Mariner,.
F.R.S.E., F.R.G.S.   With Numerous Plates, two in Colours, and Frontispiece.   5s.

Navigation: Theoretical and Practical. By D. WILSON-BARKER,.
Master Mariner, &c., and WILLIAM ALLINGHAM. 3s. 6d.

"PRECISELY the kind of work required for the New Certificates of competency in-
grades from Second Mate to extra Master. . . . Candidates will find it INVALU-
ABLE."—*Dundee Advertiser.*

Ocean Meteorology: For Officers of the Merchant Navy. By
WILLIAM ALLINGHAM, First Class Honours, Navigation, Science and Art Dep.

Practical Algebra and Trigonometry: For the Young Sailor, &c.
By RICH. C. BUCK, of the Thames Nautical Training College, H.M.S. "Worcester."

A Medical and Surgical Help for Shipmasters and Officers
IN THE MERCHANT NAVY. Including First Aid at Sea. By WM. JOHNSON SMITH,
F.R.C.S., Principal Medical Officer, Seaman's Hospital, Greenwich. With Illustra-
tions and Coloured Plates. Handsome Cloth, 6s.

"SOUND, JUDICIOUS, REALLY HELPFUL."—*The Lancet.*

LONDON: EXETER STREET, STRAND.

Griffin's Standard Publications

FOR

GEOLOGISTS, MINE-MANAGERS, AND METALLURGISTS.

Griffin's "Health" Publications.

Griffin's Chemical and Technological Publications.

In Large 8vo, with Illustrations and Folding-Plates.   10s. 6d.

BLASTING:

A Handbook for the Use of Engineers and others Engaged in Mining, Tunnelling, Quarrying, &c.

BY OSCAR GUTTMANN, ASSOC. M. INST. C.E.

*Member of the Societies of Civil Engineers and Architects of Vienna and Budapest,
Corresponding Member of the Imp. Roy. Geological Institution of Austria, &c.*

GENERAL CONTENTS.—Historical Sketch—Blasting Materials—Blasting Pow-
der—Various Powder-mixtures—Gun-cotton—Nitro-glycerine and Dynamite—
Other Nitro-compounds—Sprengel's Liquid (acid) Explosives—Other Means of
Blasting—Qualities, Dangers, and Handling of Explosives—Choice of Blasting
Materials—Apparatus for Measuring Force—Blasting in Fiery Mines—Means of
Igniting Charges—Preparation of Blasts—Bore-holes—Machine-drilling—Chamber
Mines—Charging of Bore-holes—Determination of the Charge—Blasting in Bore-
holes—Firing—Straw and Fuze Firing—Electrical Firing—Substitutes for Electrical
Firing—Results of Working—Various Blasting Operations—Quarrying—Blasting
Masonry, Iron and Wooden Structures—Blasting in earth, under water, of ice, &c.

"This ADMIRABLE work."—*Colliery Guardian.*
"Should prove a *vade-mecum* to Mining Engineers and all engaged in practical work."
—*Iron and Coal Trades Review.*

COAL-MINING (A Text-Book of):

FOR THE USE OF COLLIERY MANAGERS AND OTHERS
ENGAGED IN COAL-MINING.

BY

HERBERT WILLIAM HUGHES, F.G.S.,

Assoc. Royal School of Mines, Certificated Colliery Manager.

THIRD EDITION. *In Demy 8vo, Handsome Cloth. With very Numerous Illustrations, mostly reduced from Working Drawings.* 18s.

"The details of colliery work have been fully described, on the ground that collieries are more often made REMUNERATIVE by PERFECTION IN SMALL MATTERS than by bold strokes of engineering. . . . It frequently happens, in particular localities, that the adoption of a combination of small improvements, any of which viewed separately may be of apparently little value, turns an unprofitable concern into a paying one."—*Extract from Author's Preface.*

GENERAL CONTENTS.

Geology: Rocks –Faults—Order of Succession—Carboniferous System in Britain. **Coal**: Definition and Formation of Coal– Classification and Commercial Value of Coals. **Search for Coal**: Boring—various appliances used—Devices employed to meet Difficulties of deep Boring—Special methods of Boring—Mather & Platt's, American, and Diamond systems—Accidents in Boring—Cost of Boring—Use of Boreholes. **Breaking Ground**: Tools—Transmission of Power: Compressed Air, Electricity—Power Machine Drills—Coal Cutting by Machinery—Cost of Coal Cutting—Explosives—Blasting in Dry and Dusty Mines— Blasting by Electricity—Various methods to supersede Blasting. **Sinking**: Position, Form, and Size of shaft—Operation of getting down to "Stone-head"—Method of proceeding afterwards—Lining shafts—Keeping out Water by Tubbing—Cost of Tubbing— Sinking by Boring—Kind - Chaudron, and Lipmann methods—Sinking through Quicksands —Cost of Sinking. **Preliminary Operations**: Driving underground Roads—Supporting Roof: Timbering, Chocks or Cogs, Iron and Steel Supports and Masonry—Arrangement of Inset. **Methods of Working**: Shaft, Pillar, and Subsidence—Bord and Pillar System— Lancashire Method—Longwall Method—Double Stall Method—Working Steep Seams— Working Thick Seams—Working Seams lying near together—Spontaneous Combustion. **Haulage**: Rails—Tubs—Haulage by Horses—Self-acting Inclines—Direct-acting Haulage —Main and Tail Rope—Endless Chain - Endless Rope—Comparison. **Winding**: Pit Frames — Pulleys—Cages—Ropes—Guides—Engines—Drums—Brakes—Counterbalancing— Expansion—Condensation—Compound Engines—Prevention of Overwinding—Catches at pit top—Changing Tubs—Tub Controllers—Signalling. **Pumping**: Bucket and Plunger Pumps — Supporting Pipes in Shaft — Valves — Suspended lifts for Sinking — Cornish and Bull Engines—Davey Differential Engine—Worthington Pump—Calculations as to size of Pumps—Draining Deep Workings—Dams. **Ventilation**: Quantity of air required— Gases met with in Mines—Coal-dust—Laws of Friction—Production of Air-currents— Natural Ventilation—Furnace Ventilation—Mechanical Ventilators—Efficiency of Fans— Comparison of Furnaces and Fans—Distribution of the Air-current—Measurement of Air-currents. **Lighting**: Naked Lights – Safety Lamps— Modern Lamps — Conclusions— Locking and Cleaning Lamps— Electric Light Underground—Delicate Indicators. **Works at Surface**; Boilers—Mechanical Stoking—Coal Conveyors—Workshops. **Preparation of Coal for Market**: General Considerations—Tipplers—Screens—Varying the Sizes made by Screens—Belts— Revolving Tables—Loading Shoots—Typical Illustrations of the arrangement of Various Screening Establishments—Coal Washing—Dry Coal Cleaning –Briquettes.

"Quite THE BEST BOOK of its kind . . . as PRACTICAL in aim as a book can be . . . touches upon every point connected with the actual working of collieries. The illustrations are EXCELLENT."—*Athenæum.*
"A Text-book on Coal-Mining is a great desideratum, and Mr. HUGHES possesses ADMIRABLE QUALIFICATIONS for supplying it. . . . We cordially recommend the work."—*Colliery Guardian.*
"Mr. HUGHES has had opportunities for study and research which fall to the lot of but few men. If we mistake not, his text-book will soon come to be regarded as the STANDARD WORK of its kind."—*Birmingham Daily Gazette.*
** *Note.*—The first large edition of this work was exhausted within a few months of publication.

WORKS BY GEORGE H. HURST, F.C.S.,

Member of the Society of Chemical Industry; Lecturer on the Technology of Painters' Colours, Oils, and Varnishes, the Municipal Technical School, Manchester.

PAINTERS'
COLOURS, OILS, AND VARNISHES:

A Practical Manual.

SECOND EDITION, *Revised and Enlarged. With Numerous Illustrations. Price 12s. 6d.*

GENERAL CONTENTS.—Introductory—THE COMPOSITION, MANUFACTURE, ASSAY, and ANALYSIS of PIGMENTS, White, Red, Yellow and Orange, Green, Blue, Brown, and Black—LAKES—Colour and Paint Machinery—Paint Vehicles (Oils, Turpentine, &c., &c.)—Driers—VARNISHES.

"This useful book will prove MOST VALUABLE. We feel bound to recommend it to ALL engaged in the arts concerned."—*Chemical News.*
"A *practical* manual in every respect . . . EXCEEDINGLY INSTRUCTIVE. The section on Varnishes the most reasonable we have met with."—*Chemist and Druggist.*
"VERY VALUABLE information is given."—*Plumber and Decorator.*
"A THOROUGHLY PRACTICAL book, . . . constituting, we believe, the ONLY English work that satisfactorily treats of the manufacture of oils, colours, and pigments."—*Chemical Trades' Journal.*
"Throughout the work are scattered hints which are INVALUABLE to the intelligent reader."—*Invention.*

BY THE SAME AUTHOR.

GARMENT
DYEING AND CLEANING.

A Practical Book for Practical Men.

With Numerous Illustrations. 4s. 6d.

GENERAL CONTENTS.—Technology of the Textile Fibres—Garment Cleaning Dyeing of Textile Fabrics—Bleaching—Finishing of Dyed and Cleaned Fabrics—Scouring and Dyeing of Skin Rugs and Mats—Cleaning and Dyeing of Feathers—Glove Cleaning and Dyeing—Straw Bleaching and Dyeing—Glossary of Drugs and Chemicals—Useful Tables.

"An UP-TO-DATE hand book has long been wanted, and Mr. Hurst, who has produced several admirable works, has done nothing more complete than this. An important work, the more so that several of the branches of the craft here treated upon are almost entirely without English Manuals for the guidance of workers. The price brings it within the reach of all."—*Dyer and Calico-Printer.*
"Mr. Hurst's work DECIDEDLY FILLS A WANT . . . ought to be in the hands of EVERY GARMENT DYER and cleaner in the Kingdom."—*Textile Mercury.*

WORKS BY
ANDREW JAMIESON, M.INST.C.E., M.I.E.E., F.R.S.E.,

Professor of Electrical Engineering, The Glasgow and West of Scotland Technical College.

PROFESSOR JAMIESON'S ADVANCED MANUALS.

In Large Crown 8vo.   Fully Illustrated.

1. STEAM AND STEAM-ENGINES (A Text-Book on).

For the Use of Students preparing for Competitive Examinations. With over 200 Illustrations, Folding Plates, and Examination Papers. ELEVENTH EDITION.   Revised and Enlarged, 8/6.

"Professor Jamieson fascinates the reader by his CLEARNESS OF CONCEPTION AND SIMPLICITY OF EXPRESSION.   His treatment recalls the lecturing of Faraday."—*Athenæum.*
"The BEST BOOK yet published for the use of Students."—*Engineer.*
"Undoubtedly the MOST VALUABLE AND MOST COMPLETE Hand-book on the subject that now exists."—*Marine Engineer.*

2. MAGNETISM AND ELECTRICITY (An Advanced Text-

Book on).   Specially arranged for Advanced and "Honours" Students.

3. APPLIED MECHANICS (An Advanced Text-Book on).

Vol. I.—Comprising Part I.: The Principle of Work and its applications; Part II.: Gearing.   Price 7s. 6d.   SECOND EDITION.   [*Now ready.*

"FULLY MAINTAINS the reputation of the Author—more we cannot say."—*Pract. Engineer.*

Vol. II.—Comprising Parts III. to VI.: Motion and Energy; Strength of Materials; Graphic Statics; Hydraulics and Hydraulic Machinery.
[*In active preparation.*

PROFESSOR JAMIESON'S INTRODUCTORY MANUALS.

With numerous Illustrations and Examination Papers.

1. STEAM AND THE STEAM-ENGINE (Elementary Text-

Book on).   For First-Year Students.   FIFTH EDITION.   3/6.
"Quite the RIGHT SORT OF BOOK."—*Engineer.*
"Should be in the hands of EVERY engineering apprentice."—*Practical Engineer.*

2. MAGNETISM AND ELECTRICITY (Elementary Text-

Book on).   For First-Year Students.   FOURTH EDITION.   3/6.
"A CAPITAL TEXT-BOOK . . . The diagrams are an important feature."—*Schoolmaster.*
"A THOROUGHLY TRUSTWORTHY Text-book. . . . Arrangement as good as well can be. . . . Diagrams are also excellent. . . . The subject throughout treated as an essentially PRACTICAL one, and very clear instructions given."—*Nature.*

3. APPLIED MECHANICS (Elementary Text-Book on).

Specially arranged for First-Year Students.   SECOND EDITION.   3/6.
"Nothing is taken for granted. . . . The work has VERY HIGH QUALITIES, which may be condensed into the one word 'CLEAR.'"—*Science and Art.*

A POCKET-BOOK of ELECTRICAL RULES and TABLES.

FOR THE USE OF ELECTRICIANS AND ENGINEERS.
Pocket Size.   Leather, 8s. 6d.   *Twelfth Edition,* revised and enlarged.

LONDON : EXETER STREET, STRAND.

" The MOST VALUABLE and USEFUL WORK on Dyeing that has yet appeared in the English language . . . likely to be THE STANDARD WORK OF REFERENCE for years to come."— *Textile Mercury.*

In Two Large 8vo Volumes, 920 pp., with a SUPPLEMENTARY Volume, containing Specimens of Dyed Fabrics. Handsome Cloth, 45s.

A

MANUAL OF DYEING:

FOR THE USE OF PRACTICAL DYERS, MANUFACTURERS, STUDENTS, AND ALL INTERESTED IN THE ART OF DYEING.

BY

E. KNECHT, Ph.D., F.I.C.,   CHR. RAWSON, F.I.C., F.C.S.,

Head of the Chemistry and Dyeing Department of the Technical School, Manchester; Editor of "The Journal of the Society of Dyers and Colourists;"   Late Head of the Chemistry and Dyeing Department for the Technical College, Bradford; Member of Council of the Society of Dyers and Colourists;

And RICHARD LOEWENTHAL, Ph.D.

GENERAL CONTENTS.—Chemical Technology of the Textile Fabrics— Water—Washing and Bleaching — Acids, Alkalies, Mordants — Natural Colouring Matters—Artificial Organic Colouring Matters—Mineral Colours —Machinery used in Dyeing—Tinctorial Properties of Colouring Matters— Analysis and Valuation of Materials used in Dyeing, &c., &c.

" This MOST VALUABLE WORK . . . will be widely appreciated."—*Chemical News.*

" This authoritative and exhaustive work . . . the MOST COMPLETE we have yet seen on the subject."—*Textile Manufacturer.*

" The MOST EXHAUSTIVE and COMPLETE WORK on the subject extant."—*Textile Recorder.*

" The distinguished authors have placed in the hands of those daily engaged in the dye-house or laboratory a work of EXTREME VALUE and UNDOUBTED UTILITY . . . appeals quickly to the technologist, colour chemist, dyer, and more particularly to the rising dyer of the present generation. A book which it is refreshing to meet with."—*American Textile Record.*

GETTING GOLD:

A GOLD-MINING HANDBOOK FOR PRACTICAL MEN.

BY

J. C. F. JOHNSON, F.G.S., A.I.M.E.,

Life Member Australasian Mine-Managers' Association.

Crown 8vo, Extra.   With Illustrations.   Cloth, 3s. 6d.

"GETTING GOLD" is a compendium, in specially concrete form, of useful information respecting the processes of WINNING FROM THE SOIL and AFTER-TREATMENT of Gold and Gold ores, including some original discoveries by the Author. Practical information, original and selected, is given to Mining Company Directors, Mine Managers, Quartz Mill Operators, and Prospectors.

In "Rules of Thumb," Chapters xi. and xii , will be found a large number of useful hints on subjects directly and indirectly connected with gold mining. The Author's mining experience extends back thirty years, and it may therefore be assumed that the information, original or compiled, which the book contains, will be found both interesting and profitable to many who are engaged in that most fascinating, if not always most profitable, pursuit—"getting gold."

*** Some six years ago the Author published a small book entitled *Practical Mining*. Of this ten thousand copies were sold, nearly all in Australasia, and the work is now out of print. The London *Mining Journal* of September 9th, 1891, said of it : "We have seldom seen a book in which so much interesting matter combined with useful information is given in so small a space."

NEW VOLUME OF GRIFFIN'S MINING SERIES.

Edited by C. LE NEVE FOSTER, D.Sc., F.R.S.,

H.M. Inspector of Mines, Professor of Mining, Royal School of Mines.

Mine Accounts and Mining Book-keeping,

A Manual for the Use of Students, Managers of Metalliferous Mines and Collieries, and others interested in Mining.

With very Numerous Examples taken from the Actual Practice of leading Mining Companies throughout the world.

BY

JAMES G. LAWN, Assoc.R.S.M.,

Professor of Mining at the South African School of Mines, Capetown, Kimberley, and Johannesburg.

In Large 8vo.

LONDON : EXETER STREET, STRAND.

WORKS BY
WALTER G. M'MILLAN, F.I.C., F.C.S.,
Lecturer in Metallurgy at Mason College, Birmingham.

ELECTRIC SMELTING AND REFINING:
A PRACTICAL MANUAL OF
THE EXTRACTION AND TREATMENT OF METALS BY ELECTRICAL METHODS.
Being the "ELEKTRO-METALLURGIE" of Dr. W. BORCHERS.

Translated from the Second German Edition

BY WALTER G. M'MILLAN, F.I.C., F.C.S.

In large 8vo.   With Numerous Illustrations and Three Folding-Plates.

** THE PUBLISHERS beg to call attention to this valuable work. Dr. BORCHERS' treatise is PRACTICAL throughout. It confines itself to ONE branch of Electro-Chemistry, viz. :—ELECTROLYSIS, a subject which is daily becoming of more and more importance to the Practical Metallurgist and Manufacturer. Already in the extraction of Aluminium, the refining of Copper, the treatment of Gold and other metals, electrical processes are fast taking the place of the older methods. Dr. BORCHERS' work is acknowledged as the standard authority on the subject in Germany, and the English version, from the able pen of Mr. W. G. M'MILLAN (author of the well-known Treatise on Electro-Deposition and Electro-Plating) will, it is believed, take equal rank in English-speaking countries.

ELECTRO-METALLURGY (A Treatise on):
Embracing the Application of Electrolysis to the Plating, Depositing, Smelting, and Refining of various Metals, and to the Reproduction of Printing Surfaces and Art-Work, &c.

BY WALTER G. M'MILLAN, F.I.C., F.C.S.

With numerous Illustrations.   Large Crown 8vo.   Cloth 10s. 6d.

GENERAL CONTENTS.—Introductory—Sources of Current—General Condition to be observed in Electro-Plating—Plating Adjuncts and Disposition of Plant—Cleansing and Preparation of Work for the Depositing-Vat, and Subsequent Polishing of Plated Goods—Electro-Deposition of Copper—Electrotyping—Electro-Deposition of Silver—of Gold—of Nickel and Cobalt—of Iron—of Platinum, Zinc, Cadmium, Tin, Lead, Antimony, and Bismuth; Electro-chromy—Electro-Deposition of Alloys—Electro-Metallurgical Extraction and Refining Processes—Recovery of certain Metals from their Solutions or Waste Substances—Determination of the Proportion of Metal in certain Depositing Solutions—Appendix.

"This excellent treatise, . . . one of the BEST and MOST COMPLETE manuals hitherto published on Electro-Metallurgy."—*Electrical Review.*

"This work will be a STANDARD."—*Jeweller.*

"Any metallurgical process which REDUCES the COST of production must of necessity prove of great commercial importance. . . . We recommend this manual to ALL who are interested in the PRACTICAL APPLICATION of electrolytic processes."—*Nature.*

MACKENZIE (Thos., Master Mariner, F.R.A.S.):

PRACTICAL MECHANICS: APPLIED to the REQUIREMENTS of the SAILOR.   Crown 8vo, with numerous Illustrations.   Handsome Cloth.   3s. 6d.   [*Griffin's Nautical Series.*

" Calculated to be of GREAT PRACTICAL service."—*Fairplay.*
" THIS EXCELLENT BOOK . . . contains a LARGE AMOUNT of information."
—*Nature.*
" WELL WORTH the money . . . will be found EXCEEDINGLY HELPFUL."—
Shipping World.
" NO SHIPS' OFFICERS' BOOKCASE will henceforth be complete without
CAPTAIN MACKENZIE'S ' PRACTICAL MECHANICS.'   Notwithstanding my many
years' experience at sea, it has told me *how much more there is to acquire.*"—
(Letter to the Publishers from a Master Mariner).

MILLAR (W. J., C.E., late Secretary to the Inst.
of Engineers and Shipbuilders in Scotland):

LATITUDE AND LONGITUDE: HOW TO FIND THEM.   Crown 8vo, with Diagrams.   2s.   [*Griffin's Nautical Series.*

" CONCISELY and CLEARLY WRITTEN . . . cannot but prove an acquisition
to those studying Navigation."—*Marine Engineer.*
" Young Seamen will find it HANDY and USEFUL, SIMPLE and CLEAR."—*The
Engineer.*

SECOND EDITION.   *Enlarged, and very fully Illustrated.   Cloth,* 4s. 6d.

STEAM - BOILERS:
THEIR DEFECTS, MANAGEMENT, AND CONSTRUCTION.
BY R. D. MUNRO,
Chief Engineer of the Scottish Boiler Insurance and Engine Inspection Company.

This work, written chiefly to meet the wants of Mechanics, Engine-
keepers, and Boiler-attendants, also contains information of the first import-
ance to every user of Steam-power.   It is a PRACTICAL work written for PRAC-
TICAL men, the language and rules being throughout of the simplest nature.

"A valuable companion for workmen and engineers engaged about Steam
Boilers, ought to be carefully studied, and ALWAYS AT HAND."—*Coll. Guardian.*
" The subjects referred to are handled in a trustworthy, clear, and practical
manner. . , .   The book is VERY USEFUL, especially to steam users,
artisans, and young engineers."—*Engineer.*

BY THE SAME AUTHOR.

KITCHEN BOILER EXPLOSIONS: Why
they Occur, and How to Prevent their Occurrence?   A Practical Hand-
book based on Actual Experiment.   With Diagrams and Coloured Plate,
Price 3s.

LONDON: EXETER STREET, STRAND.

MUNRO & JAMIESON'S ELECTRICAL POCKET-BOOK.

TWELFTH EDITION, Revised and Enlarged.

A POCKET-BOOK

OF

ELECTRICAL RULES & TABLES

FOR THE USE OF ELECTRICIANS AND ENGINEERS.

BY

JOHN MUNRO, C.E., & PROF. JAMIESON, M.INST.C.E., F.R.S.E.

With Numerous Diagrams. Pocket Size. Leather, 8s. 6d.

GENERAL CONTENTS.

UNITS OF MEASUREMENT.	ELECTRO-METALLURGY.
MEASURES.	BATTERIES.
TESTING.	DYNAMOS AND MOTORS.
CONDUCTORS.	TRANSFORMERS.
DIELECTRICS.	ELECTRIC LIGHTING
SUBMARINE CABLES.	MISCELLANEOUS.
TELEGRAPHY.	LOGARITHMS.
ELECTRO-CHEMISTRY.	APPENDICES.

"WONDERFULLY PERFECT. . . . Worthy of the highest commendation we can give it."—*Electrician.*
"The STERLING VALUE of Messrs. MUNRO and JAMIESON's POCKET-BOOK."—*Electrical Review.*

MUNRO (J. M. H., D.Sc., Professor of Chemistry, Downton College of Agriculture):

AGRICULTURAL CHEMISTRY AND ANALYSIS: A PRACTICAL HAND-BOOK for the Use of Agricultural Students.

NYSTROM'S POCKET-BOOK OF MECHANICS AND ENGINEERING. Revised and Corrected by W. DENNIS MARKS, Ph.B., C.E. (YALE S.S.S.), Whitney Professor of Dynamical Engineering, University of Pennsylvania. Pocket Size. Leather, 15s. TWENTIETH EDITION, Revised and greatly enlarged.

LONDON: EXETER STREET, STRAND.

Demy 8vo, Handsome cloth, 18s.

Physical Geology and Palæontology,

ON THE BASIS OF PHILLIPS.

BY

HARRY GOVIER SEELEY, F.R.S.,

PROFESSOR OF GEOGRAPHY IN KING'S COLLEGE, LONDON.

With Frontispiece in Chromo=Lithography, and Illustrations.

"It is impossible to praise too highly the research which PROFESSOR SEELEY'S 'PHYSICAL GEOLOGY' evidences. IT IS FAR MORE THAN A TEXT-BOOK—it is a DIRECTORY to the Student in prosecuting his researches."—*Presidential Address to the Geological Society*, 1885, *by Rev. Prof. Bonney, D.Sc., LL.D., F.R.S.*

"PROFESSOR SEELEY maintains in his 'PHYSICAL GEOLOGY' the high reputation he already deservedly bears as a Teacher." — *Dr. Henry Woodward, F.R.S., in the " Geological Magazine."*

"PROFESSOR SEELEY'S work includes one of the most satisfactory Treatises on Lithology in the English language. . . . So much that is not accessible in other works is presented in this volume, that no Student of Geology can afford to be without it."—*American Journal of Engineering.*

Demy 8vo, Handsome cloth, 34s.

Stratigraphical Geology & Palæontology,

ON THE BASIS OF PHILLIPS.

BY

ROBERT ETHERIDGE, F.R.S.,

OF THE NATURAL HIST. DEPARTMENT, BRITISH MUSEUM, LATE PALÆONTOLOGIST TO THE GEOLOGICAL SURVEY OF GREAT BRITAIN, PAST PRESIDENT OF THE GEOLOGICAL SOCIETY, ETC.

With Map, Numerous Tables, and Thirty=six Plates.

. PROSPECTUS *of the above important work—perhaps the* MOST ELABORATE *of its kind ever written, and one calculated to give a new strength to the study of Geology in Britain—may be had on application to the Publishers.*

"No such compendium of geological knowledge has ever been brought together before."—*Westminster Review.*

"If PROF. SEELEY'S volume was remarkable for its originality and the breadth of its views, Mr. ETHERIDGE fully justifies the assertion made in his preface that his book differs in construction and detail from any known manual. . . . Must take HIGH RANK AMONG WORKS OF REFERENCE."—*Athenæum.*

LONDON : EXETER STREET, STRAND.

THIRD EDITION. With Folding Plates and Many Illustrations.
Large 8vo. Handsome Cloth. 36s.

ELEMENTS OF METALLURGY:

A PRACTICAL TREATISE ON THE ART OF EXTRACTING METALS FROM THEIR ORES.

BY J. ARTHUR PHILLIPS, M.INST.C.E., F.C.S., F.G.S., &c.

AND H. BAUERMAN, V.P.G.S.

GENERAL CONTENTS.

Refractory Materials.	Antimony.	Iron.
Fire-Clays.	Arsenic.	Cobalt.
Fuels, &c.	Zinc.	Nickel.
Aluminium.	Mercury.	Silver.
Copper.	Bismuth.	Gold.
Tin.	Lead.	Platinum.

*** Many NOTABLE ADDITIONS, dealing with new Processes and Developments, will be found in the Third Edition.

"Of the THIRD EDITION, we are still able to say that, as a Text-book of Metallurgy, it is THE BEST with which we are acquainted."—*Engineer*.

"The value of this work is almost *inestimable*. There can be no question that the amount of time and labour bestowed on it is enormous. . . . There is certainly no Metallurgical Treatise in the language calculated to prove of such general utility."—*Mining Journal*.

"In this most useful and handsome volume is condensed a large amount of valuable practical knowledge. A careful study of the first division of the book, on Fuels, will be found to be of great value to every one in training for the practical applications of our scientific knowledge to any of our metallurgical operations."—*Athenæum*.

"A work which is equally valuable to the Student as a Text-book, and to the practical Smelter as a Standard Work of Reference. . . . The Illustrations are admirable examples of Wood Engraving."—*Chemical News*.

POYNTING (J. H., Sc.D., F.R.S., late Fellow

of Trinity College, Cambridge; Professor of Physics, Mason College, Birmingham):
THE MEAN DENSITY OF THE EARTH: An Essay to which the Adams Prize was adjudged in 1893 in the University of Cambridge. In large 8vo, with Bibliography, Illustrations in the Text, and seven Lithographed Plates. 12s. 6d.

"An account of this subject cannot fail to be of GREAT and GENERAL INTEREST to the scientific mind. Especially is this the case when the account is given by one who has contributed so considerably as has Prof. Poynting to our present state of knowledge with respect to a very difficult subject. . . . Remarkably has Newton's estimate been verified by Prof. Poynting."—*Athenæum*.

POYNTING and THOMSON: TEXT-BOOK

OF PHYSICS. (See under *Thomson*).

WORKS BY

W. J. MACQUORN RANKINE, LL.D., F.R.S.,

Late Regius Professor of Civil Engineering in the University of Glasgow.

THOROUGHLY REVISED BY

W. J. MILLAR, C.E.,

Late Secretary to the Institute of Engineers and Shipbuilders in Scotland.

I. A MANUAL OF APPLIED MECHANICS:

Comprising the Principles of Statics and Cinematics, and Theory of Structures, Mechanism, and Machines. With Numerous Diagrams. Crown 8vo, cloth, 12s. 6d. FOURTEENTH EDITION.

II. A MANUAL OF CIVIL ENGINEERING:

Comprising Engineering Surveys, Earthwork, Foundations, Masonry, Carpentry, Metal Work, Roads, Railways, Canals, Rivers, Waterworks, Harbours, &c. With Numerous Tables and Illustrations. Crown 8vo, cloth, 16s. NINETEENTH EDITION.

III. A MANUAL OF MACHINERY AND MILLWORK:

Comprising the Geometry, Motions, Work, Strength, Construction, and Objects of Machines, &c. Illustrated with nearly 300 Woodcuts. Crown 8vo, cloth, 12s. 6d. SEVENTH EDITION.

IV. A MANUAL OF THE STEAM-ENGINE AND OTHER PRIME MOVERS:

With Numerous Tables and Illustrations, and a Diagram of the Mechanical Properties of Steam. Crown 8vo, cloth, 12s. 6d. FOURTEENTH EDITION.

V. USEFUL RULES AND TABLES:

For Architects, Builders, Engineers, Founders, Mechanics, Shipbuilders, Surveyors, &c. With APPENDIX for the use of ELECTRICAL ENGINEERS. By Professor JAMIESON, F.R.S.E. SEVENTH EDITION. 10s. 6d.

VI. A MECHANICAL TEXT-BOOK:

A Practical and Simple Introduction to the Study of Mechanics. By Professor RANKINE and E. F. BAMBER, C.E. With Numerous Illustrations. Crown 8vo, cloth, 9s. FOURTH EDITION.

** *The* "MECHANICAL TEXT-BOOK" *was designed by Professor RANKINE as an* INTRODUCTION *to the above Series of Manuals.*

PROF. RANKINE'S WORKS—(*Continued*).

VII. MISCELLANEOUS SCIENTIFIC PAPERS.

Royal 8vo. Cloth, 31s. 6d.

Part I. Papers relating to Temperature, Elasticity, and Expansion of Vapours, Liquids, and Solids. Part II. Papers on Energy and its Transformations. Part III. Papers on Wave-Forms, Propulsion of Vessels, &c.

With Memoir by Professor TAIT, M.A. Edited by W. J. MILLAR, C.E. With fine Portrait on Steel, Plates, and Diagrams.

> "No more enduring Memorial of Professor Rankine could be devised than the publication of these papers in an accessible form. . . . The Collection is most valuable on account of the nature of his discoveries, and the beauty and completeness of his analysis. . . . The Volume exceeds in importance any work in the same department published in our time "—*Architect*.

CALCAREOUS CEMENTS:

THEIR NATURE, PREPARATION, AND USES.

With some Remarks upon Cement Testing.

BY GILBERT R. REDGRAVE, ASSOC. INST. C.E.

With Illustrations. 8s. 6d.

GENERAL CONTENTS.—Introduction—Historical Review of the Cement Industry—The Early Days of Portland Cement—Composition of Portland Cement—PROCESSES OF MANUFACTURE—The Washmill and the Backs— Flue and Chamber Drying Processes—Calcination of the Cement Mixture— Grinding of the Cement—Composition of Mortar and Concrete—CEMENT TESTING — CHEMICAL ANALYSIS of Portland Cement, Lime, and Raw Materials — Employment of Slags for Cement Making — Scott's Cement, Selenitic Cement, and Cements produced from Sewage Sludge and the Refuse from Alkali Works — Plaster Cements — Specifications for Portland Cement—Appendices (Gases Evolved from Cement Works, Effects of Sea-water on Cement, Cost of Cement Manufacture, &c., &c.)

> "A work calculated to be of GREAT and EXTENDED UTILITY."—*Chemical News*.
> "INVALUABLE to the Student, Architect, and Engineer."—*Building News*.
> "A work of the GREATEST INTEREST and USEFULNESS, which appears at a very critical period of the Cement Trade."—*Brit. Trade Journal*.
> "Will be useful to ALL interested in the MANUFACTURE, USE, and TESTING of Cements."—*Engineer*.

PETROLEUM

AND ITS PRODUCTS:

A PRACTICAL TREATISE.

BY

BOVERTON REDWOOD,

F.R.S.E., F.I.C., Assoc. Inst. C.E.,

Hon. Corr. Mem. of the Imperial Russian Technical Society; Mem. of the American Chemical Society; Consulting Adviser to the Corporation of London under the Petroleum Acts, &c., &c.

ASSISTED BY GEO. T. HOLLOWAY, F.I.C., Assoc. R.C.S.,

And Numerous Contributors.

In Two Volumes, Large 8vo. Price 45s.

With Numerous Maps, Plates, and Illustrations in the Text.

GENERAL CONTENTS.

I. General Historical Account of the Petroleum Industry.

II. Geological and Geographical Distribution of Petroleum and Natural Gas.

III. Chemical and Physical Properties of Petroleum.

IV. Origin of Petroleum and Natural Gas.

V. Production of Petroleum, Natural Gas, and Ozokerite.

VI. The Refining of Petroleum.

VII. The Shale Oil and Allied Industries.

VIII. Transport, Storage, and Distribution of Petroleum.

IX. Testing of Petroleum.

X. Application and Uses of Petroleum.

XI. Legislation on Petroleum at Home and Abroad.

XII. Statistics of the Petroleum Production and the Petroleum Trade, obtained from the most trustworthy and official sources.

"The MOST COMPREHENSIVE AND CONVENIENT ACCOUNT that has yet appeared of a gigantic industry which has made incalculable additions to the comfort of civilised man. . . . The chapter dealing with the arrangement for STORAGE and TRANSPORT of GREAT PRACTICAL INTEREST. . . . The DIGEST of LEGISLATION on the subject cannot but prove of the GREATEST UTILITY."—*The Times.*

"A SPLENDID CONTRIBUTION to our technical literature."—*Chemical News.*

"This THOROUGHLY STANDARD WORK . . . in every way EXCELLENT . . . most fully and ably handled . . . could only have been produced by a man in the very exceptional position of the Author. . . . INDISPENSABLE to all who have to do with Petroleum, its APPLICATIONS, MANUFACTURE, STORAGE, or TRANSPORT."—*Mining Journal.*

"We must concede to Mr. Redwood the distinction of having produced a treatise which must be admitted to the rank of THE INDISPENSABLES. It contains THE LAST WORD that can be said about Petroleum in any of its SCIENTIFIC, TECHNICAL, and LEGAL aspects. It would be difficult to conceive of a more comprehensive and explicit account of the geological conditions associated with the SUPPLY of Petroleum and the very practical question of its AMOUNT and DURATION."—*Journ. of Gas Lighting.*

LONDON: EXETER STREET, STRAND.

Royal 8vo, Handsome Cloth, 25s.

THE STABILITY OF SHIPS.

BY

SIR EDWARD J. REED, K.C.B., F.R.S., M.P.,

KNIGHT OF THE IMPERIAL ORDERS OF ST. STANILAUS OF RUSSIA; FRANCIS JOSEPH OF AUSTRIA; MEDJIDIE OF TURKEY; AND RISING SUN OF JAPAN; VICE-PRESIDENT OF THE INSTITUTION OF NAVAL ARCHITECTS.

With numerous Illustrations and Tables.

THIS work has been written for the purpose of placing in the hands of Naval Constructors, Shipbuilders, Officers of the Royal and Mercantile Marines, and all Students of Naval Science, a complete Treatise upon the Stability of Ships, and is the only work in the English Language dealing exhaustively with the subject.

In order to render the work complete for the purposes of the Shipbuilder, whether at home or abroad, the Methods of Calculation introduced by Mr. F. K. BARNES, Mr. GRAY, M. REECH, M. DAYMARD, and Mr. BENJAMIN, are all given separately, illustrated by Tables and worked-out examples. The book contains more than 200 Diagrams, and is illustrated by a large number of actual cases, derived from ships of all descriptions, but especially from ships of the Mercantile Marine.

The work will thus be found to constitute the most comprehensive and exhaustive Treatise hitherto presented to the Profession on the Science of the STABILITY OF SHIPS.

" Sir EDWARD REED'S ' STABILITY OF SHIPS ' is INVALUABLE. In it the STUDENT, new to the subject, will find the path prepared for him, and all difficulties explained with the utmost care and accuracy ; the SHIP-DRAUGHTSMAN will find all the methods of calculation at present in use fully explained and illustrated, and accompanied by the Tables and Forms employed ; the SHIPOWNER will find the variations in the Stability of Ships due to differences in forms and dimensions fully discussed, and the devices by which the state of his ships under all conditions may be graphically represented and easily understood ; the NAVAL ARCHITECT will find brought together and ready to his hand, a mass of information which he would otherwise have to seek in an almost endless variety of publications, and some of which he would possibly not be able to obtain at all elsewhere."—*Steamship.*

"This IMPORTANT AND VALUABLE WORK . . . cannot be too highly recommended to all connected with shipping interests."—*Iron.*

"This VERY IMPORTANT TREATISE, . . . the MOST INTELLIGIBLE, INSTRUCTIVE, and COMPLETE that has ever appeared."—*Nature.*

"The volume is an ESSENTIAL ONE for the shipbuilding profession."—*Westminster Review.*

RICHMOND (H. Droop, F.C.S., Chemist to the Aylesbury Dairy Company) :

DAIRY CHEMISTRY FOR DAIRY MANAGERS : A Practical Handbook. (*Griffin's Technological Manuals.*)

LCNDON: EXETER STREET, STRAND

FOURTH EDITION, REVISED. With Additional Illustrations. Price 6s.

PRACTICAL SANITATION:

A HAND-BOOK FOR SANITARY INSPECTORS AND OTHERS INTERESTED IN SANITATION.

By GEORGE REID, M.D., D.P.H.,

Fellow of the Sanitary Institute of Great Britain, and Medical Officer, Staffordshire County Council.

With an Appendix on Sanitary Law.

By HERBERT MANLEY, M.A., M.B., D.P.H.,

Medical Officer of Health for the County Borough of West Bromwich.

GENERAL CONTENTS.—Introduction—Water Supply: Drinking Water, Pollution of Water—Ventilation and Warming — Principles of Sewage Removal — Details of Drainage ; Refuse Removal and Disposal—Sanitary and Insanitary Work and Appliances—Details of Plumbers' Work—House Construction — Infection and Disinfection — Food, Inspection of ; Characteristics of Good Meat ; Meat, Milk, Fish, &c., unfit for Human Food— Appendix : Sanitary Law ; Model Bye-Laws, &c.

"A VERY USEFUL HANDBOOK, with a very useful Appendix. We recommend it not only to SANITARY INSPECTORS, but to HOUSEHOLDERS and ALL interested in Sanitary matters."—*Sanitary Record.*

In Large 8vo, Handsome Cloth.   12s. 6d.

DISINFECTION & DISINFECTANTS

(AN INTRODUCTION TO THE STUDY OF).

Together with an Account of the Chemical Substances used as Antiseptics and Preservatives.

By SAMUEL RIDEAL, D.Sc.LOND., F.I.C., F.C.S.,

Examiner in Chemistry to the Royal College of Physicians : formerly Lecturer on Chemistry, St. George's Hospital Medical School, &c., &c.

With Folding-Plate and Illustrations of the most Approved Modern Appliances.

*** "Notwithstanding the rapid development of Sanitary Science in this country, there does not exist at the present time in the English language any book which deals exclusively with the composition of DISINFECTANTS. The present volume will, therefore, supply a want which has been felt not only by the chemist and bacteriologist, but also by those who are concerned with the practical work of disinfection. . . ."—EXTRACT FROM AUTHOR'S PREFACE.

"DR. RIDEAL'S volume is bound to prove of GREAT VALUE, both as a PRACTICAL GUIDE and as a WORK OF REFERENCE."—*Pharmaceutical Journal.*

"AN EXHAUSTIVE TREATISE, dealing with the WHOLE RANGE of the subject :—Disinfection by Heat, Chemical Disinfectants, Practical Methods, Personal Disinfection, Legal Regulations, and Methods of Analysis . . . so very well done and so USEFUL that it will be valued by ALL connected with Sanitation and Public Health."—*Chemist and Druggist.*

"A book that has long been wanted . . . will prove of VERY GREAT VALUE."—*Local Government Journal.*

LONDON : EXETER STREET, STRAND.

𝔊riffin's 𝔐etallurgical 𝔖eries.

STANDARD WORKS OF REFERENCE

FOR

Metallurgists, Mine-Owners, Assayers, Manufacturers, and all interested in the development of the Metallurgical Industries.

EDITED BY

W. C. ROBERTS-AUSTEN, C.B., F.R.S.,

CHEMIST AND ASSAYER OF THE ROYAL MINT; PROFESSOR OF METALLURGY IN THE ROYAL COLLEGE OF SCIENCE.

In Large 8vo, Handsome Cloth.   With Illustrations.

VOLUMES ALREADY PUBLISHED.

1. INTRODUCTION to the STUDY of METALLURGY.
By the EDITOR. FOURTH EDITION. 12s. 6d.

" No English text-book at all approaches this in the COMPLETENESS with which the most modern views on the subject are dealt with.   Professor Austen's volume will be INVALUABLE, not only to the student, but also to those whose knowledge of the art is far advanced."—*Chemical News.*

" INVALUABLE to the student.   .   .   .   Rich in matter not to be readily found elsewhere."—*Athenæum.*

" This volume amply realises the expectations formed as to the result of the labours of so eminent an authority.   It is remarkable for its ORIGINALITY of conception and for the large amount of information which it contains.   .   .   .   We recommend every one who desires information not only to consult, but to STUDY this work."—*Engineering.*

" Will at once take FRONT RANK as a text-book."—*Science and Art.*

" Prof. ROBERTS-AUSTEN's book marks an epoch in the history of the teaching of metallurgy in this country."—*Industries.*

2. GOLD (The Metallurgy of).   By THOS. KIRKE ROSE, D.Sc., Assoc. R.S.M., F.I.C., of the Royal Mint. SECOND EDITION, 21s.   (See p. 40).

3. IRON (The Metallurgy of).   By THOS. TURNER, Assoc. R.S.M., F.I.C., F.C.S. 16s.   (See p. 46).

Will be Published at Short Intervals.

4. STEEL (The Metallurgy of).   By F. W. HARBORD, Assoc. R.S.M., F.I.C.

5. COPPER (The Metallurgy of).   By THOS. GIBB, Assoc. Royal School of Mines.   [*At Press.*

6. METALLURGICAL MACHINERY: the Application of Engineering to Metallurgical Problems.   By HENRY CHARLES JENKINS, Wh.Sc., Assoc. R.S.M., Assoc. M. Inst. C.E., of the Royal Mint.

7. ALLOYS.   By the EDITOR.

*** Other Volumes in Preparation.

LONDON: EXETER STREET, STRAND.

SECOND EDITION, Revised and Enlarged.
In Large 8vo, Handsome cloth, 34s.

HYDRAULIC POWER

AND

HYDRAULIC MACHINERY.

BY

HENRY ROBINSON, M. INST. C.E., F.G.S.,

FELLOW OF KING'S COLLEGE, LONDON; PROF. OF CIVIL ENGINEERING,
KING'S COLLEGE, ETC., ETC.

With numerous Woodcuts, and Sixty=nine Plates.

GENERAL CONTENTS.

Discharge through Orifices—Gauging. Water by Weirs—Flow of Water through Pipes—The Accumulator—The Flow of Solids—Hydraulic Presses and Lifts—Cyclone Hydraulic Baling Press—Anderton Hydraulic Lift—Hydraulic Hoists (Lifts)—The Otis Elevator—Mersey Railway Lifts—City and South London Railway Lifts—North Hudson County Railway Elevator—Lifts for Subways—Hydraulic Ram—Pearsall's Hydraulic Engine—Pumping-Engines—Three-Cylinder Engines—Brotherhood Engine—Rigg's Hydraulic Engine—Hydraulic Capstans—Hydraulic Traversers—Movable Jigger Hoist—Hydraulic Waggon Drop—Hydraulic Jack—Duckham's Weighing Machine—Shop Tools—Tweddell's Hydraulic Rivetter—Hydraulic Joggling Press—Tweddell's Punching and Shearing Machine—Flanging Machine—Hydraulic Centre Crane—Wrightson's Balance Crane—Hydraulic Power at the Forth Bridge—Cranes—Hydraulic Coal-Discharging Machines—Hydraulic Drill—Hydraulic Manhole Cutter—Hydraulic Drill at St. Gothard Tunnel—Motors with Variable Power—Hydraulic Machinery on Board Ship—Hydraulic Points and Crossings—Hydraulic Pile Driver—Hydraulic Pile Screwing Apparatus—Hydraulic Excavator—Ball's Pump Dredger—Hydraulic Power applied to Bridges—Dock-gate Machinery—Hydraulic Brake—Hydraulic Power applied to Gunnery—Centrifugal Pumps—Water Wheels—Turbines—Jet Propulsion—The Gerard-Barré Hydraulic Railway—Greathead's Injector Hydrant—Snell's Hydraulic Transport System—Greathead's Shield—Grain Elevator at Frankfort—Packing—Power Co-operation—Hull Hydraulic Power Company—London Hydraulic Power Company—Birmingham Hydraulic Power System—Niagara Falls—Cost of Hydraulic Power—Meters—Schönheyder's Pressure Regulator—Deacon's Waste-Water Meter.

" A Book of great Professional Usefulness."—Iron.

*** The SECOND EDITION of the above important work has been thoroughly revised and brought up to date. Many new full-page Plates have been added—the number being increased from 43 in the First Edition to 69 in the present. Full Prospectus, giving a description of the Plates, may be had on application to the Publishers.

LONDON: EXETER STREET, STRAND.

GRIFFIN'S METALLURGICAL SERIES.

THE METALLURGY OF GOLD.

BY

T. KIRKE ROSE, D.Sc., Assoc.R.S.M., F.I.C.,

Assistant Assayer of the Royal Mint.

SECOND EDITION. Revised and partly Re-written. Including the most recent Improvements in the Cyanide Process, and a new Chapter on Economic Considerations (Management, Cost, Output, &c.). With Frontispiece and additional Illustrations. Large 8vo. Handsome Cloth. 21s.

LEADING FEATURES.

1. Adapted for all who are interested in the Gold Mining Industry, being free from technicalities as far as possible ; of special value to those engaged in the industry—viz., mill-managers, reduction-officers, &c.

2. The whole ground implied by the term "Metallurgy of Gold" has been covered with equal care ; the space is carefully apportioned to the various branches of the subject, according to their relative importance.

3. The MACARTHUR-FORREST CYANIDE PROCESS is fully described for the first time. By this process over £2,000,000 of gold per annum (at the rate of) is now being extracted, or nearly one-tenth of the total world's production. The process, introduced in 1887, has only had short newspaper accounts given of it previously. The chapters have been submitted to, and revised by, Mr. MacArthur, and so freed from all possible inaccuracies.

4. Among other new processes not previously described in a text-book are— (1) The modern barrel chlorination process, practised with great success in Dakota, where the Black Hills district is undergoing rapid development owing to its introduction. (2) New processes for separating gold from silver—viz., the new Gutzkow process, and the Electrolytic process ; the cost of separation is reduced by them by one-half.

5. A new feature is the description of EXACT METHODS employed in particular extraction works—Stamp-batteries of South Africa, Australia, New Zealand, California, Colorado, and Dakota; Chlorination works also, in many parts of the world ; Cyanide works of S. Africa and New Zealand. These accounts are of special value to practical men.

6. The bibliography is the first made since 1882.

"Dr. ROSE gained his experience in the Western States of America, but he has secured details of gold-working from ALL PARTS of the world, and these should be of GREAT SERVICE to practical men. . . . The four chapters on *Chlorination*, written from the point of view alike of the practical man and the chemist, TEEM WITH CONSIDERATIONS HITHERTO UNRECOGNISED, and constitute an addition to the literature of Metallurgy, which will prove to be of classical value."—*Nature.*

"The most complete description of the chlorination process which has yet been published. —*Mining Journal.*

NEW AND IMPORTANT WORK.

Companion-Volume to MM. Knecht and Rawson's "Dyeing."

TEXTILE PRINTING:

A PRACTICAL MANUAL.

Including the Processes Used in the Printing of
COTTON, WOOLLEN, and SILK FABRICS.

BY

C. F. SEYMOUR ROTHWELL, F.C.S.,

Mem. Soc. of Chemical Industries; late Lecturer at the Municipal Technical School, Manchester.

In Large 8vo, with Illustrations in the text and Specimens of Printed
Patterns.

SCHWACKHÖFER and BROWNE:

FUEL AND WATER: A Manual for Users of Steam and Water.
By Prof. FRANZ SCHWACKHÖFER of Vienna, and WALTER
R. BROWNE, M.A., C.E., late Fellow of Trinity College, Cambridge.
Demy 8vo, with Numerous Illustrations, 9/.

GENERAL CONTENTS.—Heat and Combustion—Fuel, Varieties of—Firing Arrange-
ments : Furnace, Flues, Chimney — The Boiler, Choice of -- Varieties — Feed-water
Heaters—Steam Pipes—Water : Composition, Purification—Prevention of Scale, &c., &c.

"The Section on Heat is one of the best and most lucid ever written."—*Engineer.*
"Cannot fail to be valuable to thousands using steam power."—*Railway Engineer.*

SHELTON-BEY (W. Vincent, Foreman to the
Imperial Ottoman Gun Factories, Constantinople) :

THE MECHANIC'S GUIDE : A Hand-Book for Engineers and
Artizans. With Copious Tables and Valuable Recipes for Practical Use.
Illustrated. *Second Edition.* Crown 8vo. Cloth, 7/6.

LONDON : EXETER STREET, STRAND.

Thirteenth Edition.   Price 21s.

Demy 8vo, Cloth.   With Numerous Illustrations, reduced from Working Drawings.

A MANUAL OF
MARINE ENGINEERING:

COMPRISING THE DESIGNING, CONSTRUCTION, AND WORKING OF MARINE MACHINERY.

By A. E. SEATON, M. Inst. C. E., M. Inst. Mech. E., M.Inst.N.A.

GENERAL CONTENTS.

Part I.—Principles of Marine Propulsion.

Part II.—Principles of Steam Engineering.

Part III.—Details of Marine Engines: Design and Cal-

culations for Cylinders, Pistons, Valves, Expansion Valves, &c.

Part IV.—Propellers.

Part V.—Boilers.

Part VI.—Miscellaneous.

** The THIRTEENTH EDITION includes a Chapter on WATER-TUBE BOILERS, with Illustrations of the leading Types.

" In the three-fold capacity of enabling a Student to learn how to design, construct, and work a Marine Steam-Engine, Mr. Seaton's Manual has NO RIVAL."—*Times.*
" The important subject of Marine Engineering is here treated with the THOROUGH-NESS that it requires.   No department has escaped attention. . . . Gives the results of much close study and practical work."—*Engineering.*
" By far the BEST MANUAL in existence. . . . Gives a complete account of the methods of solving, with the utmost possible economy, the problems before the Marine Engineer."—*Athenæum.*
" The Student, Draughtsman, and Engineer will find this work the MOST VALUABLE HANDBOOK of Reference on the Marine Engine now in existence."—*Marine Engineer.*

THIRD EDITION.   With Diagrams.   Pocket-Size, Leather.   8s. 6d.

A POCKET-BOOK OF
MARINE ENGINEERING RULES AND TABLES,

FOR THE USE OF

Marine Engineers, Naval Architects, Designers, Draughtsmen, Superintendents and Others.

BY

A. E. SEATON, M. I. C. E., M. I. Mech. E., M. I. N.'A.,

AND

H. M. ROUNTHWAITE, M.I.Mech.E., M.I.N.A.

" ADMIRABLY FULFILS its purpose."—*Marine Engineer.*

LONDON: EXETER STREET, STRAND.

WORKS BY A. HUMBOLDT SEXTON, F.I.C., F.C.S.,

Professor of Metallurgy in the Glasgow and West of Scotland Technical College.

In Large Crown 8vo, Handsome Cloth, 6s.

ELEMENTARY METALLURGY

(A TEXT-BOOK OF).

Including the Author's PRACTICAL LABORATORY COURSE.

With Numerous Illustrations.

GENERAL CONTENTS.—Introduction—Properties of the Metals—Combustion—Fuels—Refractory Materials—Furnaces—Occurrence of the Metals in Nature—Preparation of the Ore for the Smelter—Metallurgical Processes—Iron : Preparation of Pig Iron—Malleable Iron—Steel—Mild Steel—Copper—Lead—Zinc and Tin—Silver—Gold—Mercury—Alloys—Applications of ELECTRICITY to Metallurgy—LABORATORY COURSE WITH NUMEROUS PRACTICAL EXERCISES.

" The volume before us FULLY ENHANCES and confirms PROF. SEXTON'S reputation. . . . Just the kind of work for Students COMMENCING the study of Metallurgy, or for ENGINEERING Students requiring a GENERAL KNOWLEDGE of it, or for ENGINEERS in practice who like a HANDY WORK OF REFERENCE. To all three classes we HEARTILY commend the work."—*Practical Engineer.*

" EXCELLENTLY got-up and WELL-ARRANGED. . . . Iron and copper well explained by EXCELLENT diagrams showing the stages of the process from start to finish. . . . The most NOVEL chapter is that on the many changes wrought in Metallurgical Methods by ELECTRICITY."—*Chemical Trade Journal.*

" Possesses the GREAT ADVANTAGE of giving a COURSE OF PRACTICAL WORK."—*Mining Journal.*

Sexton's (Prof.) Outlines of Quantitative Analysis.

FOR THE USE OF STUDENTS.

With Illustrations. FOURTH EDITION. Crown 8vo, Cloth, 3s.

" A COMPACT LABORATORY GUIDE for beginners was wanted, and the want has been WELL SUPPLIED. . . . A good and useful book."—*Lancet.*

Sexton's (Prof.) Outlines of Qualitative Analysis.

FOR THE USE OF STUDENTS.

With Illustrations. THIRD EDITION. Crown 8vo, Cloth, 3s. 6d.

" The work of a thoroughly practical chemist."—*British Medical Journal.*
" Compiled with great care, and will supply a want."—*Journal of Education.*

LONDON: EXETER STREET, STRAND.

WORKS BY PROF. ROBERT H. SMITH, Assoc.M.I.C.E.,

M.I.M.E., M.I.El.E., M.Fed.I.Mi.E., Whit. Sch., M.Ord.Meiji.

MEASUREMENT CONVERSIONS

(English and French):

28 GRAPHIC TABLES OR DIAGRAMS.

Showing at a glance the MUTUAL CONVERSION of MEASUREMENTS
in DIFFERENT UNITS

**Of Lengths, Areas, Volumes, Weights, Stresses, Densities, Quantities
of Work, Horse Powers, Temperatures, &c.**

For the use of Engineers, Surveyors, Architects, and Contractors.

In 4to, Boards.   7s. 6d.

** Prof. SMITH's CONVERSION-TABLES form the most unique and comprehensive collection ever placed before the profession. By their use much time and labour will be saved, and the chances of error in calculation diminished. It is believed that henceforth no Engineer's Office will be considered complete without them.

" The work is INVALUABLE."—*Colliery Guardian.*

"Ought to be in EVERY office where even occasional conversions are required. . . . Prof. SMITH's TABLES form very EXCELLENT CHECKS on results. . . . A VERY USEFUL and good set of diagrams."—*Electrical Review.*

"Prof. Smith deserves the hearty thanks, not only of the ENGINEER, but of the COMMERCIAL WORLD, for having smoothed the way for the ADOPTION of the METRIC SYSTEM of MEASUREMENT, a subject which is now assuming great importance as a factor in maintaining our HOLD upon FOREIGN TRADE. There can be no doubt that the antiquated system of Weights and Measures used in this country is doomed to be superseded by the much simpler method of DECIMAL MEASUREMENT. The sooner this is recognised, the better."—*The Machinery Market.*

THE CALCULUS FOR ENGINEERS AND PHYSICISTS,

Applied to Technical Problems.

WITH EXTENSIVE

CLASSIFIED REFERENCE LIST OF INTEGRALS.

By PROF. ROBERT H. SMITH.

ASSISTED BY

ROBERT FRANKLIN MUIRHEAD,

M.A., B.Sc. (Glasgow), B.A. (Cambridge),
Formerly Clark Fellow of Glasgow University, and Lecturer on Mathematics at Mason College.

In Crown 8vo, extra, with Diagrams and Folding-Plate.

LONDON : EXETER STREET, STRAND.

By PROFESSORS J. J. THOMSON & POYNTING.

In Large 8vo. Fully Illustrated.

A TEXT-BOOK OF PHYSICS:

COMPRISING

PROPERTIES OF MATTER; HEAT; SOUND AND LIGHT; MAGNETISM AND ELECTRICITY.

BY

J. H. POYNTING,		J. J. THOMSON,
SC.D., F.R.S.,	AND	M.A., F.R.S.,
Late Fellow of Trinity College, Cambridge; Professor of Physics, Mason College, Birmingham.		Fellow of Trinity College, Cambridge; Prof. of Experimental Physics in the University of Cambridge.

THIRD EDITION, *Revised and Enlarged. Pocket-Size, Leather, also for Office Use, Cloth,* 12s. 6d.

BOILERS, MARINE AND LAND;
THEIR CONSTRUCTION AND STRENGTH.

A HANDBOOK OF RULES, FORMULÆ, TABLES, &c., RELATIVE TO MATERIAL, SCANTLINGS, AND PRESSURES, SAFETY VALVES, SPRINGS, FITTINGS AND MOUNTINGS, &c.

For the Use of all Steam=Users.

By T. W. TRAILL, M. INST. C. E., F. E. R. N.,
Late Engineer Surveyor-in-Chief to the Board of Trade.

⁎ To THE THIRD EDITION MANY NEW TABLES have been added.

"Very unlike any of the numerous treatises on Boilers which have preceded it. . . . Really useful. . . . Contains an ENORMOUS QUANTITY OF INFORMATION arranged in a very convenient form. . . . Those who have to design boilers will find that they can settle the dimensions for any given pressure with almost no calculation with its aid. . . . A MOST USEFUL VOLUME . . . supplying information to be had nowhere else."—*The Engineer.*

"As a handbook of rules, formulæ, tables, &c., relating to materials, scantlings, and pressures, this work will prove MOST USEFUL. The name of the Author is a sufficient guarantee for its accuracy. It will save engineers, inspectors, and draughtsmen a vast amount of calculation."—*Nature.*

"By such an authority cannot but prove a welcome addition to the literature of the subject. . . . We can strongly recommend it as being the MOST COMPLETE, eminently practical work on the subject."—*Marine Engineer.*

"To the engineer and practical boiler-maker it will prove INVALUABLE. The tables in all probability are the most exhaustive yet published. . . . Certainly deserves a place on the shelf in the drawing office of every boiler shop."—*Practical Engineer.*

GRIFFIN'S METALLURGICAL SERIES.

THE METALLURGY OF IRON.

BY

THOMAS TURNER, Assoc.R.S.M., F.I.C.,

Director of Technical Instruction to the Staffordshire County Council.

IN LARGE 8VO, HANDSOME CLOTH, WITH NUMEROUS ILLUSTRATIONS
(MANY FROM PHOTOGRAPHS).   PRICE 16s.

GENERAL CONTENTS.

Early History of Iron.	Slags and Fluxes of Iron Smelting.
Modern History of Iron.	Properties of Cast Iron.
The Age of Steel.	Foundry Practice.
Chief Iron Ores.	Wrought Iron.
Preparation of Iron Ores.	Indirect Production of Wrought
The Blast Furnace.	Iron.
The Air used in the Blast Furnace.	The Puddling Process.
Reactions of the Blast Furnace.	Further Treatment of Wrought
The Fuel used in the Blast Furnace.	Iron.

Corrosion of Iron and steel.

"A MOST VALUABLE SUMMARY of useful knowledge relating to every method and stage in the manufacture of cast and wrought iron down to the present moment . . . particularly rich in chemical details. . . . An EXHAUSTIVE and REALLY NEEDED compilation by a MOST CAPABLE and THOROUGHLY UP-TO-DATE metallurgical authority."—*Bulletin of the American Iron and Steel Association.*

"This is A DELIGHTFUL BOOK, giving, as it does, reliable information on a subject becoming every day more elaborate. . . . The account of the chief iron ores is, like the rest of this work, RICH in detail. . . . Foundry Practice has been made the subject of considerable investigation by the author, and forms an interesting and able chapter."—*Colliery Guardian.*

"Mr. Turner's work comes at an opportune moment and in answer to a REAL DEMAND. . . . A THOROUGHLY USEFUL BOOK, which brings the subject UP TO DATE. The author has produced an EMINENTLY READABLE BOOK. . . . Whatever he describes, he describes well. . . . There is much in the work that will be of GREAT VALUE to those engaged in the iron industry."—*Mining Journal.*

IN PREPARATION.

COMPANION-VOLUME ON

THE   METALLURGY   OF   STEEL.

By F. W. HARBORD, Assoc.R.S.M., F.I.C.

KNOW YOUR OWN SHIP.

By THOMAS WALTON, NAVAL ARCHITECT.

SPECIALLY ARRANGED TO SUIT THE REQUIREMENTS OF

Ships' Officers, Shipowners, Superintendents, Draughtsmen, Engineers, and others. This work explains, in a simple manner, such important subjects as :—

Displacement, Deadweight, Tonnage, Freeboard, Moments, uoyancy, Strain, Structure, Stability, Rolling, Ballasting, Loading, Shifting Cargoes, Admission of Water, Sail Area, &c., &c.

SECOND EDITION.   *With Numerous Illustrations.   Handsome Cloth.   ·5s.*

" The little book will be found EXCEEDINGLY HANDY by most officers and officials connected with shipping. . . . Mr. Walton's work will obtain LASTING SUCCESS, because of its unique fitness for those for whom it has been written."—*Shipping World.*

" An EXCELLENT WORK, full of solid instruction and INVALUABLE to every officer of the Mercantile Marine who has his profession at heart."—*Shipping.*

" Not one of the 242 pages could well be spared. It will admirably fulfil its purpose . . . useful to ship owners, ship superintendents, ship draughtsmen, and all interested in shipping."—*Liverpool Journal of Commerce.*

" A mass of VERY USEFUL INFORMATION, accompanied by diagrams and illustrations, is given in a compact form."—*Fairplay.*

" A large amount of MOST USEFUL INFORMATION is given in the volume. The book is certain to be of great service to those who desire to be thoroughly grounded in the subject of which it treats."—*Steamship.*

" We have found no one statement that we could have wished differently expressed. The matter has, so far as clearness allows, been admirably condensed, and is simple enough to be understood by every seaman."—*Marine Engineer.*

GAS AND OIL ENGINES:

An Introductory Text-book on the Theory, Design, Construction, and Testing of Internal Combustion Engines without Boiler.

FOR THE USE OF STUDENTS.

BY

PROF. W. H. WATKINSON, WHIT. SCH., M.INST.MECH.E..

Glasgow and West of Scotland Technical College.

In Crown 8vo, extra, with Numerous Illustrations.   [Shortly.

LONDON: EXETER STREET, STRAND.

WORKS BY DR. ALDER WRIGHT, F.R.S.

FIXED OILS, FATS, BUTTERS, AND WAXES:

THEIR PREPARATION AND PROPERTIES,

And the Manufacture therefrom of Candles, Soaps, and Other Products.

BY

C. R. ALDER WRIGHT, D.Sc., F.R.S.,

Late Lecturer on Chemistry, St. Mary's Hospital School ; Examiner in "Soap" to the City and Guilds of London Institute.

In Large 8vo.   Handsome Cloth.   With 144 Illustrations.   28s.

" Dr. WRIGHT'S work will be found ABSOLUTELY INDISPENSABLE by every Chemist. TEEMS with information valuable alike to the Analyst and the Technical Chemist."— *The Analyst.*

"Will rank as the STANDARD ENGLISH AUTHORITY on OILS and FATS for many years to come."—*Industries and Iron.*

SECOND EDITION. With very Numerous Illustrations.   Handsome Cloth, 6s. Also Presentation Edition, Gilt and Gilt Edges, 7s. 6d.

THE THRESHOLD OF SCIENCE:

Simple and Amusing Experiments (over 400) in Chemistry and Physics.

⁎⁎⁎ To the NEW EDITION has been added an excellent chapter on the Systematic Order in which Class Experiments should be carried out for Educational purposes.

"Any one who may still have doubts regarding the value of Elementary Science as an organ of education will speedily have his doubts dispelled, if he takes the trouble to understand the methods recommended by Dr. Alder Wright. The Additions to the New Edition will be of great service to all who wish to use the volume, not merely as a ' play-book,' but as an instrument for the TRAINING of the MENTAL FACULTIES."—*Nature.*

" Step by step the learner is here gently guided through the paths of science, made easy by the perfect knowledge of the teacher, and made flowery by the most striking and curious experiments. Well adapted to become the TREASURED FRIEND of many a bright and promising lad."—*Manchester Examiner.*

LONDON: EXETER STREET, STRAND.

Engineering Drawing and Design

(A TEXT-BOOK OF).

SECOND EDITION. In Two Parts, Published Separately.

VOL. I.—PRACTICAL GEOMETRY, PLANE, AND SOLID. 3s.

VOL. II.—MACHINE AND ENGINE DRAWING AND DESIGN. 4s. 6D.

BY

SIDNEY H. WELLS, WH.SC.,

A.M.INST.C.E., A.M.INST.MECH.E.,

Principal of, and Head of the Engineering Department in, the Battersea Polytechnic Institute
formerly of the Engineering Departments of the Yorkshire College, Leeds;
and Dulwich College, London.

*With many Illustrations, specially prepared for the Work, and numerous
Examples, for the Use of Students in Technical Schools and Colleges.*

" A THOROUGHLY USEFUL WORK, exceedingly well written. For the many Examples and
Questions we have nothing but praise."—*Nature.*
" A CAPITAL TEXT-BOOK, arranged on an EXCELLENT SYSTEM, calculated to give an intelligent
grasp of the subject, and not the mere faculty of mechanical copying. . . . Mr. Wells shows
how to make COMPLETE WORKING-DRAWINGS, discussing fully each step in the design."—*Electrical
Review.*
" The first book leads EASILY and NATURALLY towards the second, where the technical pupil
is brought into contact with large and more complex designs."—*The Schoolmaster.*

Electrical Measurements & Instruments.

A Practical Hand-book of Testing for the Electrical Engineer.

BY CHARLES H. YEAMAN,

Assoc. Inst. E.E., formerly Electrical Engineer to the Corporation of Liverpool.

The author has had an extensive experience in the use of Electrical Measuring Instruments of
various types in the Laboratory, Testroom, and Workshop.
The following subjects of practical importance are dealt with—

SUPPLY METERS. | POTENTIOMETERS.
GALVANOMETERS.

The testing of Supply Meters is a subject upon which very little has been published. The
Potentiometer has not been systematically treated in any recent book on electrical measurement,
and although Galvanometers are considered by many writers, there still remains the want for a
connected description of the different types and their uses.
The Electrical Engineer is particularly concerned with—

ALTERNATING CURRENT MEASUREMENTS.
FAULTY LOCATION.
DYNAMO AND TRANSFORMER EFFICIENCY TESTS.
LOW RESISTANCES. | MEDIUM RESISTANCES.
HIGH RESISTANCES.
BATTERY TESTING. | MAGNETIC TESTS.
DESCRIPTION OF TEST-ROOMS AND ELECTRO-TECHNICAL LABORATORIES.

This work is written by an Electrical Engineer for Electrical Engineers. It forms an excellent
text-book for students proceeding to the "Electrical Instruments" section of the Honours Grade
Examinations in Electrical Engineering of the City and Guilds of London Institute.
The majority of the illustrations have not been published before, and have been specially
prepared for this work.

LONDON: EXETER STREET, STRAND.

Fourteenth Annual Issue. Handsome cloth, 7s. 6d.

THE OFFICIAL YEAR-BOOK

OF THE

SCIENTIFIC AND LEARNED SOCIETIES OF GREAT BRITAIN AND IRELAND.

COMPILED FROM OFFICIAL SOURCES.

Comprising (together with other Official Information) LISTS of the PAPERS read during 1896 before all the LEADING SOCIETIES throughout the Kingdom engaged in the following Departments of Research :—

§ 1. Science Generally : *i.e.*, Societies occupying themselves with several Branches of Science, or with Science and Literature jointly.
§ 2. Mathematics and Physics.
§ 3. Chemistry and Photography.
§ 4. Geology, Geography, and Mineralogy.
§ 5. Biology, including Microscopy and Anthropology.

§ 6. Economic Science and Statistics.
§ 7. Mechanical Science and Architecture.
§ 8. Naval and Military Science.
§ 9. Agriculture and Horticulture.
§ 10. Law.
§ 11. Literature.
§ 12. Psychology.
§ 13. Archæology.
§ 14. Medicine.

Of the TWELFTH ISSUE (for 1895), *The Engineer* says—"Every year of publication of this book has ADDED to the PROOFS of its USEFULNESS. . . . The Year-books of past years form a VERY HANDY CONSECUTIVE INDEX of the work done by the Societies."

" The YEAR-BOOK OF SOCIETIES is a Record which ought to be of the greatest use for the progress of Science."—*Sir Lyon Playfair, F.R.S., K.C.B., M.P., Past-President of the British Association.*

" It goes almost without saying that a Handbook of this subject will be in time one of the most generally useful works for the library or the desk."—*The Times.*

"British Societies are now well represented in the 'Year-Book of the Scientific and Learned Societies of Great Britain and Ireland.'"—(Art. "Societies" in New Edition of "Encyclopædia Britannica," vol. xxii.)

Copies of the FIRST ISSUE, giving an Account of the History, Organization, and Conditions of Membership of the various Societies, and forming the groundwork of the Series, may still be had, price 7/6. *Also Copies of the following Issues.*

The YEAR-BOOK OF SOCIETIES forms a complete INDEX TO THE SCIENTIFIC WORK of the year in the various Departments. It is used as a ready HANDBOOK in all our great SCIENTIFIC CENTRES, MUSEUMS, and LIBRARIES throughout the Kingdom, and has become an INDISPENSABLE BOOK OF REFERENCE to every one engaged in Scientific Work.

FIRST SERIES—THIRTY-SIXTH EDITION.
SECOND SERIES—TENTH EDITION.

MANY THOUGHTS OF MANY MINDS:

A Treasury of Reference, consisting of Selections from the Writings of the most Celebrated Authors. FIRST AND SECOND SERIES.

COMPILED AND ANALYTICALLY ARRANGED

By HENRY SOUTHGATE.

———◆———

Each Series is complete in itself, and sold separately.

Presentation Edition, Cloth and Gold, .   .   . 12s. 6d. each volume.
Library Edition, Half Bound, Roxburghe, .   . 14s.   ,,
    Do.,   Morocco Antique,   .   .   . 21s.   ,,

In Square 8vo, elegantly printed on toned paper.